Homework 8

Exercise 1. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Check that the increments of the five processes below are again distributed as those of a standard Brownian motion:

a) $B_t^{(1)} = -B_t,\, t \in \mathbb{R}_+$ ($\leftrightarrow$ "spatial" symmetry of the Brownian motion).

b) Let $T \in \mathbb{R}_+$: $B_t^{(2)} = B_{t+T} - B_T,\, t \in \mathbb{R}_+$ ($\leftrightarrow$ stationarity).

c) Let $T \in \mathbb{R}_+$: $B_t^{(3)} = B_T - B_{T-t},\, t \in [0,T]$ ($\leftrightarrow$ time-reversal).

d) Let $a > 0$: $B_t^{(4)} = \frac{1}{\sqrt{a}} B_{at},\, t \in \mathbb{R}_+$ ($\leftrightarrow$ scaling law).

e) $B_t^{(5)} = t B_{\frac{1}{t}},\, t > 0$ and $B_0^{(5)} = 0$ ($\leftrightarrow$ time inversion).

Remark: These five processes are actually all standard Brownian motions (proof not required).

Exercise 2. Among the symmetric functions $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ below, determine which are positive semi-definite. When this is the case, describe the centered Gaussian process $(X_t,\, t \in \mathbb{R}_+)$ with covariance $K(t,s) = \mathbb{E}(X_t\, X_s)$.

a) $K^{(1)}(t,s) = t \wedge s [= \min(t,s)]$.

Hint: Show by induction that if $t_1 \geq \ldots \geq t_n$, then $\sum_{i,j=1}^n c_i\, c_j\, (t_i \wedge t_j) \geq (c_1 + \ldots + c_n)^2\, t_n \geq 0$.

b) $K^{(2)}(t,s) = g(t)\, g(s)$, where $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous.

c) $K^{(3)}(t,s) = t + s$.

d) $K^{(4)}(t,s) = e^{-ts} - 1$.

e) $K^{(5)}(t,s) = e^{ts} - 1$.

Remark: For the last two cases, do not try describing the Gaussian process with covariance $K(t,s)$ (if it exists!).

Exercise 3. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Compute the mean and the covariance of the following two processes:

a) $M_t = B_t^2 - t, \quad t \in \mathbb{R}_+$.

b) $N_t = \exp(B_t - \frac{t}{2}), \quad t \in \mathbb{R}_+$.

Is any of these two processes Gaussian?