Homework 3

Exercise 1. (reverse Chebychev's inequality)
Let X be a square-integrable random variable such that $X \geq 0$ a.s. Let also $0 \leq t < \mathbb{E}(X)$.

a) Show that

$$\mathbb{P}(\{X > t\}) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)}.$$

Hint: Use Cauchy-Schwarz's inequality.

b) *Application:* Check that the above inequality holds in the particular case $X \sim \mathcal{P}(\lambda)$ and $t = 0$.

Exercise 2. Let X be a centered random variable with variance σ^2. Using Chebychev's inequality, show that:

a) $\mathbb{P}(\{|X| \geq a\}) \leq \dfrac{\sigma^2}{a^2}$ and $\mathbb{P}(\{|X| \geq a\}) \leq \dfrac{2\sigma^2}{a^2 + \sigma^2}$.

b) $\mathbb{P}(\{X \geq a\}) \leq \dfrac{\sigma^2}{a^2 + \sigma^2}$ (use $\psi(x) = (x + b)^2$ with $b \geq 0$, then minimize over b).

Exercise 3. Show the weak law of large numbers stated in class.

Hint: Use Chebychev's inequality.

Exercise 4. (Why it is not a good idea to play at roulette too many times)
On a classical roulette game with 38 numbers (including the 0 and the 00), a player bets uniquely on red, 361 times in a row. At each turn, he bets exactly one franc (he therefore wins one franc if red comes out and loses one franc if this is not the case). Assuming that the roulette wheel is balanced and that the turns are independent from each other, give a rough estimate of:

a) the player's fortune at the end of the 361 games;

b) the probability that he has actually won some money.

Hints: - Remember that the number 0 and 00 are neither red nor black on a classical roulette.
- Use the central limit theorem.