Homework 2

Exercise 1. Check that the distributions below are well defined distributions and compute, when they exist, the mean and the variance of these distributions.

A) Discrete distributions:

a) Bernoulli $\mathcal{B}i(1,p)$, $p \in [0,1]$: $\mathbb{P}(X=1) = p$, $\mathbb{P}(X=0) = 1-p$.

b) binomial $\mathcal{B}i(n,p)$, $n \geq 1$, $p \in [0,1]$: $\mathbb{P}(X=k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k\,(1-p)^{n-k}, \quad 0 \leq k \leq n.$

c) Poisson $\mathcal{P}(\lambda)$, $\lambda > 0$: $\mathbb{P}(X=k) = \frac{\lambda^k}{k!}\,e^{-\lambda}, \quad k \geq 0.$

B) Continuous distributions:

d) uniform $\mathcal{U}([a,b])$, $a < b$: $f_X(x) = \frac{1}{b-a}\,1_{[a,b]}(x), \quad x \in \mathbb{R}.$

e) Gaussian $\mathcal{N}(\mu,\sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\exp(-\frac{(x-\mu)^2}{2\sigma^2}), \quad x \in \mathbb{R}.$

f) Cauchy $\mathcal{C}(\lambda)$, $\lambda > 0$: $f_X(x) = \frac{1}{\pi}\,\frac{\lambda}{\lambda^2+x^2}, \quad x \in \mathbb{R}.$

g) exponential $\mathcal{E}(\lambda)$, $\lambda > 0$: $f_X(x) = \lambda\,e^{-\lambda x}, \quad x \in \mathbb{R}_+.$

h) Gamma $\Gamma(t,\lambda)$, $t, \lambda > 0$: $f_X(x) = \frac{(\lambda x)^{t-1}\,\lambda\,e^{-\lambda x}}{\Gamma(t)}, \quad x \in \mathbb{R}_+, \quad$ où $\Gamma(t) := \int_0^\infty dx\,x^{t-1}\,e^{-x}.$

Exercise 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using only the axioms given in the definition of a probability measure, show the following properties:

a) $\mathbb{P}(A) \leq \mathbb{P}(B)$, if $A \subset B,\ A, B \in \mathcal{F}$.

b) $\mathbb{P}(\cup_{n=1}^\infty B_n) \leq \sum_{n=1}^\infty \mathbb{P}(B_n)$, if $(B_n)_{n=1}^\infty \subset \mathcal{F}$.

c) $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$, if $A \subset B,\ A, B \in \mathcal{F}$.

d) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, if $A \in \mathcal{F}$.

e) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, if $A, B \in \mathcal{F}$.

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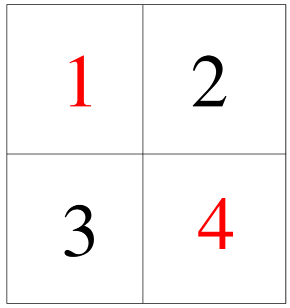
Exercise 3. One considers the following simplified roulette game:



a) Let us assume equal probabilities for all numbers.

Is the family of events "red", "odd" and "1 or 2" independent?

Are these events 2-by-2 independent?

b) Consider the same question in the case where the roulette is biased as follows:

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = 0.3, \quad \mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = 0.2.$$

Exercise 4. Let X be a centered Gaussian random variable of variance σ^2. Compute:

a) $\mathbb{E}(X^4)$.

b) $\mathbb{E}(\exp(X))$.

c) $\mathbb{E}(\exp(-X^2))$.