Homework 11

Exercise 1. a) Let X and Y be two square-integrable random variables. Verify that

$$\mathrm{Cov}(X,Y) = \frac{1}{4}\left(\mathrm{Var}(X+Y) - \mathrm{Var}(X-Y)\right).$$

Remark: This identity is known as the *polarization identity.*

The quadratic *covariation* of two continuous square-integrable martingales $(M_t,\, t \in \mathbb{R}_+)$ and $(N_t,\, t \in \mathbb{R}_+)$ adapted to the same filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ has been defined in the class as the unique process $(\langle M,N\rangle_t,\, t \in \mathbb{R}_+)$ which is adapted, continuous and with bounded variation, such that $\langle M,N\rangle_0 = 0$ a.s. and $(M_t\, N_t - \langle M,N\rangle_t,\, t \in \mathbb{R}_+)$ is a martingale.

An alternate definition of quadratic covariation is the following:

$$\langle M,N\rangle_t = \frac{1}{4}\left(\langle M+N\rangle_t - \langle M-N\rangle_t\right), \quad t \in \mathbb{R}_+.$$

b) Using this definition, show that the process $(M_t\, N_t - \langle M,N\rangle_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t, t \in \mathbb{R}_+)$.

c) Using this definition again, show that the process $(\langle M,N\rangle_t,\, t \in \mathbb{R}_+)$ has bounded variation.

Exercise 2. (Wiener's integral) Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Let also $T > 0$, $f : [0,T] \to \mathbb{R}$ be a continuous function on $[0,T]$ and $t_i = \frac{iT}{2^n}$ for $i = 1, \ldots, 2^n$. One defines

$$f^{(n)}(t) = \sum_{i=1}^{2^n} f(t_{i-1})\, 1_{]t_{i-1},t_i]}(t) \quad \text{and} \quad (f^{(n)} \cdot B)_T \equiv \int_0^T f^{(n)}(s)\, dB_s = \sum_{i=1}^{2^n} f(t_{i-1})\,(B_{t_i} - B_{t_{i-1}}),$$

Similarly, one defines

$$(f^{(n)} \cdot B)_t \equiv \int_0^t f^{(n)}(s)\, dB_s = \sum_{i=1}^{k-1} f(t_{i-1})\,(B_{t_i} - B_{t_{i-1}}) + f(t_{k-1})\,(B_t - B_{t_{k-1}}),$$

if $t \in [0,T]$ is such that $t_{k-1} < t \le t_k$.

a) By the class, it is known that $((f^{(n)} \cdot B)_t,\, t \in [0,T])$ is a continuous square-integrable martingale (why exactly?). In this particular context, express the isometry relation.

b) Show that $((f^{(n)} \cdot B)_t,\, t \in [0,T])$ is a Gaussian process, and compute its mean and covariance.

c) Show that $((f^{(n)} \cdot B)_t,\, t \in [0,T])$ is a process with independent increments.

Important remark: It is a fact that there exists a process $(f \cdot B)$ such that for all $t \in [0,T]$, $(f^{(n)} \cdot B)_t \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} (f \cdot B)_t$ and that all the above properties remain valid for the limiting object $(f \cdot B)_t$.

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Exercise 3. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and let $(H_t,\, t \in \mathbb{R}_+)$ be a continuous process adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Let also $\varepsilon \in [0,1]$, $T \in \mathbb{R}_+$ and $t_i = \frac{iT}{2^n}$ for $i = 1, \ldots, 2^n$. One further defines (assuming that the limit exists in probability):

$$(H \cdot B)_T^{(\varepsilon)} = \lim_{n \to \infty} \sum_{i=1}^{2^n} \big((1-\varepsilon)\, H_{t_{i-1}} + \varepsilon\, H_{t_i}\big)\, (B_{t_i} - B_{t_{i-1}}),$$

Show that

$$(H \cdot B)_T^{(\varepsilon)} = (H \cdot B)_T^{(0)} + \varepsilon\, \langle H, B \rangle_T,$$

where

$$\langle H, B \rangle_T = \lim_{n \to \infty} \sum_{i=1}^{2^n} (H_{t_i} - H_{t_{i-1}})\, (B_{t_i} - B_{t_{i-1}}).$$

Remark: The above definition corresponds to Ito's integral when $\varepsilon = 0$, and to Fisk-Stratonovic's integral when $\varepsilon = 1/2$.