Quiz: Solutions

1. a) false, b) true, c) true, d) true, e) no.

2. a) $\mathbb{E}((X+Y)^2|Y) = \mathbb{E}(X^2) + 2\,\mathbb{E}(X)\,Y + Y^2$

b) $\mathbb{E}((X+Y)^2|Y) = 2$

c) An explicit computation can be done, or one can use the following more general argument: by symmetry, $\mathbb{E}(X|X+Y) = \mathbb{E}(Y|X+Y)$, and $\mathbb{E}(X|X+Y)+\mathbb{E}(Y|X+Y) = \mathbb{E}(X+Y|X+Y) = X+Y$, so $\mathbb{E}(X(X|X+Y) = \mathbb{E}(Y|X+Y) = \frac{X+Y}{2}$.

3. a) no, b) no, c) yes, d) $\mathbb{E}(M_n) = 0$, $\mathrm{Cov}(M_n, M_m) = 2$ if $m = n$, $= 1$ if $|n-m| = 1$, $= 0$ otherwise.

4. a) yes (apply Jensen)

b) yes: $\mathbb{E}(S_{n+1}^4 \,|\, \mathcal{F}_n) = S_n^4 + 6S_n^2 + 1$, so $\mathbb{E}(S_{n+1}^4 - (n+1) \,|\, \mathcal{F}_n) = (S_n^4 - n) + 6S_n^2 \geq S_n^4 - n$.

c) From b), $\mathbb{E}(S_{n+1}^4) = \mathbb{E}(S_n^2) + 6n + 1$, so by induction, $\mathbb{E}(S_n^4) = 3n^2 + 2n - 1$

d) $\lim_{n\to\infty} \frac{\mathbb{E}(S_n^4)}{n^2} = 3$. This couls also be deduced directly form the central limit theorem, which states that $S_n/\sqrt{n}$ converges in distribution to a $\mathcal{N}(0,1)$ random variable, whose fourth moment is equal to 3.

5. a) no, b) yes, c) no, d) yes, e) no.

6. a) $2B_t - B_s \sim \mathcal{N}(0, 4t - 3s)$, b) yes, c) yes, d) yes, e) yes, f) no.

7. a) yes, b) NO!, c) no, d) yes, e) $\mathrm{Var}(M_t - M_s) = t^2 - s^2$. NB: actually, $M_t = \int_0^t \sqrt{2s}\, dB_s$

8. $X_t = M_t + V_t$, where $M_t = 1 + \int_0^t aX_s\, dB_s$ is a martingale and $V_t = \int_0^t (b + a^2/2) X_s\, ds$ is a process with bounded variation.

b) $b + a^2/2 \geq 0$, c) $b + a^2/2 = 0$, d) $\langle X \rangle_t = a^2 \int_0^t X_s^2\, ds = a^2 \int_0^t \exp(2aB_s - a^2 s)\, ds$.