Solutions 2

1. a) $\mathbb{E}(X) = p$, $\mathrm{Var}(X) = p(1-p)$.
b) $\mathbb{E}(X) = np$, $\mathrm{Var}(X) = np(1-p)$.
c) $\mathbb{E}(X) = \lambda$, $\mathrm{Var}(X) = \lambda$.
d) $\mathbb{E}(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$.
e) $\mathbb{E}(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$.
f) $\mathbb{E}(X)$ are $\mathrm{Var}(X)$ are not defined.
g) $\mathbb{E}(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.
h) $\mathbb{E}(X) = \frac{t}{\lambda}$, $\mathrm{Var}(X) = \frac{t}{\lambda^2}$.

2. a) use $B = A \cup (B \backslash A)$, where A and $B \backslash A$ are disjoint.

b) use $\cup_{n=1}^\infty B_n = \cup_{n=1}^\infty A_n$, where $A_n = B_n \backslash (B_1 \cup \ldots \cup B_{n-1})$; the A_n are disjoint, so by axiom (ii) and a),

$$\mathbb{P}(\cup_{n=1}^\infty B_n) = \mathbb{P}(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mathbb{P}(A_n) \leq \sum_{n=1}^\infty \mathbb{P}(B_n)$$

c) use again $B = A \cup (B \backslash A)$.

d) use $\Omega = A \cup A^c$ and $\mathbb{P}(\Omega) = 1$.

e) use $A \cup B = A \cup (B \backslash (A \cap B))$, where A and $B \backslash (A \cap B)$ are disjoint, along with c).

3. a) We have $\mathbb{P}(\{i\}) = 0.25$ for all i, so $\mathbb{P}(\{\text{red}\}) = \mathbb{P}(\{1,4\}) = 0.5$, $\mathbb{P}(\{\text{odd}\}) = \mathbb{P}(\{1,3\}) = 0.5$, $\mathbb{P}(\{1 \text{ or } 2\}) = \mathbb{P}(\{1,2\}) = 0.5$, as well as

$$\begin{aligned}
\mathbb{P}(\{\text{red}\} \cap \{\text{odd}\}) &= \mathbb{P}(\{1\}) = 0.25 = \mathbb{P}(\{\text{red}\})\,\mathbb{P}(\{\text{odd}\}), \\
\mathbb{P}(\{\text{red}\} \cap \{1 \text{ or } 2\}) &= \mathbb{P}(\{1\}) = 0.25 = \mathbb{P}(\{\text{red}\})\,\mathbb{P}(\{1 \text{ or } 2\}), \\
\mathbb{P}(\{\text{odd}\} \cap \{1 \text{ or } 2\}) &= \mathbb{P}(\{1\}) = 0.25 = \mathbb{P}(\{\text{odd}\})\,\mathbb{P}(\{1 \text{ or } 2\}).
\end{aligned}$$

So “red”, “odd’ and “1 or 2” are 2-by-2 independent, but they are not independent as a family of 3 events, since

$$\begin{aligned}
\mathbb{P}(\{\text{red}\} \cap \{\text{odd}\} \cap \{1 \text{ or } 2\}) &= \mathbb{P}(\{1\}) = 0.25 \\
&\neq 0.125 = \mathbb{P}(\{\text{red}\})\,\mathbb{P}(\{\text{odd}\})\,\mathbb{P}(\{1 \text{ or } 2\}).
\end{aligned}$$

b) In this case, $\mathbb{P}(\{\text{red}\}) = 0.5$, $\mathbb{P}(\{\text{odd}\}) = 0.5$, $\mathbb{P}(\{1 \text{ or } 2\}) = 0.6$, et

$$\begin{aligned}
\mathbb{P}(\{\text{red}\} \cap \{\text{odd}\}) &= \mathbb{P}(\{1\}) = 0.3 \neq \mathbb{P}(\{\text{red}\})\,\mathbb{P}(\{\text{odd}\}), \\
\mathbb{P}(\{\text{red}\} \cap \{1 \text{ or } 2\}) &= \mathbb{P}(\{1\}) = 0.3 = \mathbb{P}(\{\text{red}\})\,\mathbb{P}(\{1 \text{ or } 2\}), \\
\mathbb{P}(\{\text{odd}\} \cap \{1 \text{ or } 2\}) &= \mathbb{P}(\{1\}) = 0.3 = \mathbb{P}(\{\text{odd}\})\,\mathbb{P}(\{1 \text{ or } 2\}).
\end{aligned}$$

So “red” and “1 or 2”, as well as “odd” and “1 or 2” are independent, but not “red” et “odd”. From this, one deduces that the family “red”, “odd” and “1 or 2” cannot be independent.

4. a) By integration by parts, one obtains:

$$\mathbb{E}(X^4) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x^3 \cdot x \exp\left(-\frac{x^2}{2\sigma^2}\right) \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} 3x^2 \cdot \sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) \, dx = 3\sigma^4$$

b)

$$\mathbb{E}(\exp(X)) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(x - \frac{x^2}{2\sigma^2}\right) \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(\frac{-(x-\sigma^2)^2}{2\sigma^2} + \frac{\sigma^2}{2}\right) \, dx = \exp\left(\frac{\sigma^2}{2}\right).$$

c)

$$\mathbb{E}(\exp(-X^2)) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-x^2\left(1 + \frac{1}{2\sigma^2}\right)\right) \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{\frac{\pi}{1 + \frac{1}{2\sigma^2}}} = \frac{1}{\sqrt{2\sigma^2 + 1}}.$$