Quiz

1. Let $X \sim \mathcal{N}(0, \sigma^2)$, with $0 < \sigma < 1/\sqrt{2}$.

a) Compute $\mathbb{E}(\exp(X))$ and $\mathbb{E}(\exp(X^2))$.

Let X_1, X_2 be independent and such that $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$, with $0 < \sigma_1, \sigma_2 < 1/\sqrt{2}$.

b) What is the value of $\mathbb{E}(\exp(X_1 X_2))$?

$b1)\quad \exp(\sigma_1^2\, \sigma_2^2/2) \qquad b2)\quad \exp((\sigma_1^2 + \sigma_2^2)/2)$

$b3)\quad \dfrac{1}{\sqrt{1 - \sigma_1^2\, \sigma_2^2}} \qquad b4)\quad \dfrac{1}{\sqrt{1 - (\sigma_1^2 + \sigma_2^2)}}$

c) Consider now $\sigma_1 = \sigma_2 = \sigma$. Which of the following is the largest: $\mathbb{E}(\exp(X^2))$ or $\mathbb{E}(\exp(X_1 X_2))$?

2. Let M be a process such that $M_t > 0$ for all $t \in \mathbb{R}_+$ and let A be a deterministic increasing process such that $A_0 = 0$. Let also $X_t = 1/M_t$, $Y_t = M_t + A_t$ and $Z_t = M_t\, A_t$. What can you say on the processes X, Y and Z if

a) M is a martingale?

b) M is a submartingale?

c) M is a supermartingale?

3. Let B be a standard Brownian motion and let

$$\tau = \inf\{t > 0 \ : \ B_t \geq 1\}.$$

a) Does there exist a constant $K > 0$ such that $\tau \leq K$ a.s.? Is $\mathbb{P}(\tau < \infty) = 1$?

b) Is it true that $\mathbb{E}(B_\tau) = \mathbb{E}(B_0)$?

c) Among the following inequalities, which are true?

$c1)\quad \mathbb{P}(\tau \geq t) \geq \mathbb{P}(B_t \geq 1) \qquad c2)\quad \mathbb{P}(\tau \geq t) \geq \mathbb{P}(B_t \leq 1)$
$c3)\quad \mathbb{P}(\tau \geq t) \leq \mathbb{P}(B_t \geq 1) \qquad c4)\quad \mathbb{P}(\tau \geq t) \leq \mathbb{P}(B_t \leq 1)$
$c5)\quad \mathbb{P}(\tau \leq t) \geq \mathbb{P}(B_t \geq 1) \qquad c6)\quad \mathbb{P}(\tau \leq t) \geq \mathbb{P}(B_t \leq 1)$
$c7)\quad \mathbb{P}(\tau \leq t) \leq \mathbb{P}(B_t \geq 1) \qquad c8)\quad \mathbb{P}(\tau \leq t) \leq \mathbb{P}(B_t \leq 1)$

Let now

$$\tau' = \inf\{t > 0 \ : \ B_t^2 \geq 1\}.$$

d) Does there exist a constant $K > 0$ such that $\tau' \leq K$ a.s.? Is $\mathbb{P}(\tau' < \infty) = 1$?

e) Is it true that $\mathbb{E}(B_{\tau'}) = \mathbb{E}(B_0)$?

Let finally X be the process defined as $X_t = \exp(B_t - t/2)$ and let

$$\tau'' = \inf\{t > 0 \ : \ X_t \geq 2\}.$$

f*) Does there exist a constant $K > 0$ such that $\tau'' \leq K$ a.s.? Is $\mathbb{P}(\tau'' < \infty) = 1$?

g*) Is it true that $\mathbb{E}(X_{\tau''}) = \mathbb{E}(X_0)$?

h*) Compute $\mathbb{P}(\tau'' < \infty)$.

please turn the page.

4. Let B be a standard Brownian motion and let

$$X_t = \int_0^t \sqrt{2s}\, dB_s, \quad Y_t = \int_0^t X_s\, dB_s, \quad Z_t = \int_0^t s\, dB_s.$$

a) Which of the processes X, Y, Z are martingales?

b) Which of the processes X, Y, Z are Gaussian?

c) Compute $\mathbb{E}(X_t^2)$, $\mathbb{E}(Y_t^2)$ and $\mathbb{E}(Z_t^2)$.

d) Is the process $(Y_t^2 - \mathbb{E}(Y_t^2),\, t \in \mathbb{R}_+)$ a martingale?

e) Is the process $(Z_t^2 - \mathbb{E}(Z_t^2),\, t \in \mathbb{R}_+)$ a martingale?

f) Is the process $(Y_t^2 - Z_t^2,\, t \in \mathbb{R}_+)$ a martingale?

5. Let $(B^{(1)}, B^{(2)})$ be a standard two-dimensional Brownian motion. For which of the following two-dimensional processes $(X^{(1)}, X^{(2)})$ does there exist a probability measure under which $(X^{(1)}, X^{(2)})$ is a two-dimensional martingale?

a) $dX_t^{(1)} = X_t^{(1)}\,(dB_t^{(1)} + dB_t^{(2)}), \quad dX_t^{(2)} = X_t^{(2)}\, dB_t^{(2)}$.

b) $dX_t^{(1)} = X_t^{(1)}\,(dB_t^{(1)} + dB_t^{(2)}), \quad dX_t^{(2)} = X_t^{(2)}\,(dB_t^{(1)} + dB_t^{(2)})$.

c) $dX_t^{(1)} = X_t^{(1)}\, dB_t^{(1)}, \quad dX_t^{(2)} = X_t^{(2)}\, dB_t^{(1)}$.

d) $dX_t^{(1)} = X_t^{(1)}\, dB_t^{(2)}, \quad dX_t^{(2)} = X_t^{(2)}\, dB_t^{(1)}$.

6. Let B be a standard Brownian motion and M be the strong solution of the SDE

$$dM_t = \exp(M_t)\, dB_t, \quad M_0 = 0.$$

Fact. Such a process exists up to a (possibly random and finite) explosion time τ.

a) For $n \geq 1$, let $\tau_n = \inf\{t > 0 \,:\, |M_t| \geq n\}$. What can you say on the process $(M_{t \wedge \tau_n},\, t \in \mathbb{R}_+)$?

b) Compute the SDE satisfied by the process $X_t = \exp(M_t)$.

7. Let B be a standard Brownian motion. Among the following functions f, which are such that the process $f(t, B_t)$ is a (local) martingale?

a) $f(t,x) = x^2 - t$	*b)* $f(t,x) = x - t/2$
c) $f(t,x) = x^2/t$	*d)* $f(t,x) = x^3 - 3tx$
e) $f(t,x) = x^4 - 6tx^2 + 3t^2$	*f)* $f(t,x) = x^4 - 4t^2x^2 + t^4$
g) $f(t,x) = \exp(x^2 - t)$	*h)* $f(t,x) = \exp(x - t/2)$

8. Who is going to win Roland-Garros' final?

Answers session on Monday, June 8, at 2:00 PM, in room INR 113.

Exam on Wednesday, June 17, at 2:00 PM, in room INM 200.
Authorized material: 4 handwritten one-sided A4 pages.
Exam duration: 4 hours.