## Homework 9

**Exercise 1.** Let us consider the two-dimensional Black-Scholes equation

$$\begin{cases} dX_t^{(1)} = X_t^{(1)} \left( \mu_1 \, dt + \sigma_{11} \, dB_t^{(1)} + \sigma_{12} \, dB_t^{(2)} \right), \quad X_0^{(1)} = x_0^{(1)}, \\ \\ dX_t^{(2)} = X_t^{(2)} \left( \mu_2 \, dt + \sigma_{21} \, dB_t^{(1)} + \sigma_{22} \, dB_t^{(2)} \right), \quad X_0^{(2)} = x_0^{(2)}, \end{cases}$$

where $x_0^{(1)}, x_0^{(2)} > 0$, $\underline{B} = (B^{(1)}, B^{(2)})$ is a standard two-dimensional Brownian motion, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} > 0$.

a) Give a necessary and sufficient condition on $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ so that the diffusion $\underline{X} = (X^{(1)}, X^{(2)})$ is non-degenerate on the set $D = \{\underline{x} \in \mathbb{R}^2 : x_1 \neq 0 \text{ and } x_2 \neq 0\}$.

b) When this condition is satisfied, compute the martingale $M$ which serves as a basis for the definition of the probability measure $\widetilde{\mathbb{P}}_T$ under which the processes $X^{(1)}, X^{(2)}$ are martingales (up to time $T$).

**Exercise 2.** (Poisson's equation)
Let $D$ be an open and bounded domain in $\mathbb{R}^n$ and $\partial D$ be its (smooth) boundary. We assume that the following result is known: given $g \in C(D)$, there exists a unique $u \in C^2(D)$ satisfying

$$\begin{cases} \frac{1}{2} \Delta u(\underline{x}) = -g(\underline{x}), & \underline{x} \in D, \\ u(\underline{x}) = 0, & \underline{x} \in \partial D. \end{cases}$$

Let now $(\underline{B}_t^{\underline{x}}, t \in \mathbb{R}_+)$ be an $n$-dimensional Brownian motion starting at point $\underline{x} \in D$ at time $t = 0$ and let

$$\tau = \inf\{t > 0 : \underline{B}_t^{\underline{x}} \notin D\}$$

be the first exit time of $\underline{B}^{\underline{x}}$ from $D$ (notice that $\tau$ depends on the starting point $\underline{x}$).

a) Show that

$$u(\underline{x}) = \mathbb{E}\left( \int_0^\tau g(\underline{B}_s^{\underline{x}}) \, ds \right), \quad \underline{x} \in D.$$

b) Let now $D = B(0,1)$ be the unit ball in $\mathbb{R}^n$ and let us define

$$u(\underline{x}) = \frac{1 - \|\underline{x}\|^2}{n}.$$

Obviously, $u(\underline{x}) = 0$ on $\partial D = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| = 1\}$. Find the function $g$ such that $u$ satisfies the above PDE.

c) Application: deduce the value of $\mathbb{E}(\tau)$ as a function of $\underline{x}$.

d) In the special case where $\underline{x} = 0$, how does this value behave as the dimension $n$ increases?

e) Can you find an intuitive explanation for this last observation?