Homework 8

Exercise 1. (Bessel's process)
Let $n \geq 2$ and $(\underline{B}_t,\, t \in \mathbb{R}_+)$ be a standard n-dimensional Brownian motion.

a) Show that the process $(W_t,\, t \in \mathbb{R})$ defined as

$$W_t = \sum_{i=1}^{n} \int_0^t \frac{B_s^{(i)}}{\|\underline{B}_s\|}\, dB_s^{(i)}, \quad t \in \mathbb{R}_+,$$

is a standard (one-dimensional) Brownian motion.

b) Using Ito-Doeblin's formula, show that the process $R_t = \|\underline{B}_t\|$ is solution of the SDE

$$dR_t = \frac{n-1}{2R_t}\, dt + dW_t$$

Remark: The function $f(\underline{x}) = \|\underline{x}\|$ is not differentiable in $\underline{x} = 0$, but this is not an issue here, as one can show that for $n \geq 2$, the Brownian motion does not come back too often to the position $x = 0$ (unlike in the one-dimensional case).

Exercise 2. Let $x_0 \in \mathbb{R}$, $\sigma > 0$ and $(B_t, t \in \mathbb{R}_+)$ be a standard (one-dimensional) Brownian motion. In this exercise, we look for the solution of the two-dimensional SDE

$$\begin{cases} dX_t = Y_t\, dt, & X_0 = x_0, \\ dY_t = -X_t\, dt + \sigma\, dB_t, & Y_0 = 0. \end{cases}$$

a) Rewrite the above equation in the form

$$d\underline{Z}_t = A\, \underline{Z}_t\, dt + \Sigma\, d\underline{B}_t, \quad \underline{Z}_0 = \underline{z}_0,$$

where $\underline{Z}_t = (X_t, Y_t)$ [column vector], A is a 2×2 matrix, Σ is a 2×1 matrix and $\underline{B}_t = B_t$.

b) Compute the (matrix-valued) solution Φ_t of the homogeneous equation

$$d\Phi_t = A\, \Phi_t\, dt, \quad \Phi_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hint: Remember that $\cos(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}$, $\sin(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!}$.
Notice also that $A^2 = -I$.

c) Show that $(\Phi_t)^{-1} = \Phi_{-t}$ and $\Phi_t\, \Phi_s = \Phi_{t+s}$.

d) Deduce the solution $\underline{Z}_t = (X_t, Y_t)$ of the original SDE.

e) Check that indeed, $Y_t = \dfrac{dX_t}{dt}$.