Homework 7

Exercise 1. Let $x_0 \in \mathbb{R}$, $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $f, g : \mathbb{R} \to \mathbb{R}$ be Lipschitz and bounded. Let also $(X_t,\, t \in \mathbb{R}_+)$ be the strong solution of the SDE

$$dX_t = f(X_t)\, dt + g(X_t)\, dB_t, \quad X_0 = x_0,$$

and $A : C^2(\mathbb{R}) \to C(\mathbb{R})$ the linear differential operator defined as

$$Av(x) = f(x)\, v'(x) + \frac{1}{2}\, g(x)^2\, v''(x), \quad \text{for } x \in \mathbb{R} \text{ and } v \in C^2(\mathbb{R}).$$

Let now $v \in C^2(\mathbb{R})$ be such that v' is bounded. Show successively that

a) $v(X_t) - v(x_0) - \int_0^t Av(X_s)\, ds$ is a martingale.

b) $\lim_{t \downarrow 0} \frac{1}{t}\, \mathbb{E}(v(X_t) - v(x_0)) = Av(x_0)$.

c) $\lim_{t \downarrow 0} \frac{1}{t}\, \mathbb{E}(X_t - x_0) = f(x_0)$.

d) $\lim_{t \downarrow 0} \frac{1}{t}\, \mathbb{E}((X_t - x_0)^2) = g(x_0)^2$.

Exercise 2. Let $x_0 > 0$, $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t, x), Lipshitz in x and bounded. Let also $(X_t,\, t \in \mathbb{R}_+)$ be the strong solution of the SDE

$$dX_t = f(t, X_t)\, dt + dB_t, \quad X_0 = x_0.$$

Let $T > 0$ and $C_T = \max(X_T - K, 0)$ be a European call option on the stock X.

a) Compute the premium $c(t, X_t)$ of this option as a function of the time t and the stock price X_t.

b) Compute the hedging strategy $(\phi_t, t \in [0, T])$ of this option.

NB: Try to obtain formulas as explicit as possible.

Exercise 3. In the classical Black-Scholes model with time-independent coefficients, compute the hedging strategy $(\phi_t, t \in [0, T])$ of a call option with strike price K and maturity T. Again, try deriving the most explicit formula for this strategy.