Homework 6

Exercise 1. (the "true" Feynman-Kac formula)
In this exercise, we are looking for a probabilistic representation of the solution of the following PDE:

$$\begin{cases} u'_t(t,x) + \frac{1}{2}\, u''_{xx}(t,x) + c(t,x)\, u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ u(T,x) = h(x), & x \in \mathbb{R}, \end{cases}$$

where $T > 0$, $h \in C(\mathbb{R})$ and $c : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

a) Let u be the solution of the above equation and $(B_t^{t_0,x_0},\ t \geq t_0)$ be a Brownian motion starting from $x_0 \in \mathbb{R}$ at time $t_0 \in [0,T[$. We define the processes

$$C_t = \exp\left(\int_{t_0}^t c(s, B_s^{t_0,x_0})\, ds \right) \quad \text{and} \quad M_t = C_t\, u(t, B_t^{t_0,x_0}), \quad \text{for } t \in [t_0, T].$$

Using successively the integration by parts and Ito-Doeblin's formulas, show that the process $(M_t, t \in [t_0,T])$ is a martingale.

b) Deduce an expression for $u(t_0, x_0)$.

Exercise 2. Let us consider the following PDE:

$$\begin{cases} u'_t(t,x) = -a\, x\, u'_x(t,x) + \frac{1}{2}\, u''_{xx}(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u_0 \in C(\mathbb{R})$.

a) For a given $T > 0$ and $x \in \mathbb{R}$, find the process $(X_t^x,\ t \in [0,T])$ associated to the above equation and deduce a probabilistic representation for the solution $u(T,x)$ of the equation.

b) Given what you already know on the distribution of the process X^x, deduce an expression as explicit as possible for $u(T,x)$. Consider the particular case

$$u_0(y) = \max(y, 0).$$

Exercise 3. Let us consider the following PDE:

$$\begin{cases} u'_t(t,x) = -\frac{x}{t}\, u'_x(t,x) + \frac{1}{2}\, u''_{xx}(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u_0 \in C(\mathbb{R})$.

a) For a given $T > 0$ and $x \in \mathbb{R}$, find the process $(X_t^x,\ t \in [0,T])$ associated to the above equation and deduce a probabilistic representation for the solution $u(T,x)$ of the equation.

b) What's wrong here?