Homework 5

Exercise 1. (Black-Scholes model with time-dependent coefficients)
Let $\mu, \sigma : \mathbb{R}_+ \to \mathbb{R}$ be two continuous functions such that $|\mu(t)| \leq K_1$ and $0 < K_2 \leq |\sigma(t)| \leq K_1$, for all $t \in \mathbb{R}_+$. Let also $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion. We consider the following SDE:

$$dX_t = \mu(t)\, X_t\, dt + \sigma(t)\, X_t\, dB_t, \quad X_0 = x_0 > 0.$$

a) Explain why we know in advance that there exists a unique strong solution $(X_t,\, t \in \mathbb{R}_+)$ to the above equation. Find this solution.

b) For fixed $T > 0$, describe the probability measure $\widetilde{\mathbb{P}}_T$ under which $(X_t, t \in [0, T])$ is a martingale. Find also the process $(\widetilde{B}_t, t \in [0, T])$ which is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$. Express X in terms of $\widetilde{B}$.

c) X represents the evolution of the price of a stock with drift $\mu(t)$ and volatility $\sigma(t)$ at time t, in a market where the risk-free rate is equal to zero. Evaluate the premium c_0 one has to pay at time $t = 0$ in order to buy a European call option on the stock X with maturity T and strike price K. Push the computation as far as you can, in order to get the most possible explicit formula for the premium.

Exercise 2. Coming back to the Black-Scholes model seen in class with time-*independent* coefficients, evaluate the premium C_t of a call option at time $t < T$. This premium will now be given as a function of both $T - t$ (the time-to-maturity) and X_t (the stock price at time t).

Hint: As already mentioned, the premium C_t is now an ($\mathcal{F}_t$-measurable) random variable depending on the value of X_t; the expectations are therefore to be replaced by conditional expectations.

Exercise 3. Again under the Black-Scholes model with time-independent coefficients, compute the premium at time $t = 0$ of an option whose payoff at time T is

$$C_T = 1_{X_T \geq K}.$$

Such options are called "digital" options.