Homework 4

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X, Z be two random variables defined on this probability space, where Z is a $\mathcal{N}(0,1)$ random variable, independent of X.

a) Let $Y = \exp(-XZ - X^2/2)$. Compute $\mathbb{E}(Y|X)$.

b) Let $\widetilde{\mathbb{P}}$ be a new probability measure on $(\Omega, \mathcal{F})$ defined as

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}(1_A Y), \quad A \in \mathcal{F}.$$

Check that $\widetilde{\mathbb{P}}(\Omega) = 1$ (the other properties of a probability measure can be checked similarly).

c) The expectation with respect to $\widetilde{\mathbb{P}}$ of a random variable U such that $\mathbb{E}(|UY|) < \infty$ is given by $\widetilde{\mathbb{E}}(U) = \mathbb{E}(UY)$. Show that for any continuous bounded function $g : \mathbb{R} \to \mathbb{R}$,

$$\widetilde{\mathbb{E}}(g(X+Z)) = \int_{\mathbb{R}} dy\, g(y)\, \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2},$$

which shows that under the probability measure $\widetilde{\mathbb{P}}$, the random variable $X+Z$ is a $\mathcal{N}(0,1)$ random variable.

Exercise 2. (A Brownian motion writes your name in finite time with positive probability!)
One can show (but this is not required here) the following reasonable statement: if $(B_t,\, t \in \mathbb{R}_+)$ is a standard Brownian motion, then for any fixed $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0 \le t \le T} |B_t| \le \varepsilon \right) > 0.$$

What is more surprising is the following statement, which you are asked to prove (using the previous one). Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a deterministic and continuously differentiable function such that $g(0) = 0$. Then for any $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0 \le t \le T} |B_t - g(t)| \le \varepsilon \right) > 0.$$

Remarks: - This property remains true for a Brownian motion in two dimensions (which we haven't seen yet). If you therefore think of the function $g : \mathbb{R}_+ \to \mathbb{R}^2$ as the one writing your name, you reach the conclusion mentioned at the beginning of this exercise.
- One can even show that the above probability can be made equal to one, provided one can choose the scale at which one observes the Brownian motion!

Exercise 3. Let $(X_t,\, t \in [0,1])$ be the Brownian bridge considered in Homework 3, Exercise 2. Following what has been done in class, let $(M_t,\, t \in [0,1])$ be the martingale used to define a new probability measure $\widetilde{\mathbb{P}}_1$ under which X should be a martingale. Show nevertheless that there does not exist a contant $K > 0$ such that

$$\langle M \rangle_t \le Kt, \quad \forall t \in [0,1].$$

This is an indication (not a proof) that it is actually impossible in this case to find a probability measure under which X is a martingale. Do you see another reason for this?