Homework 3

Exercise 1. (Black-Scholes equation)
Let $\mu \in \mathbb{R}$, $\sigma > 0$ be fixed and $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. One considers the following stochastic differential equation (SDE):

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dB_t, \quad t > 0, \qquad X_0 = x_0 > 0.$$

Find the solution $(X_t,\, t \in \mathbb{R}_+)$ of this equation.

Hint: Set first $X_t = \Phi_t\, Z_t$, where Φ_t is solution of $d\Phi_t = \mu\, \Phi_t\, dt$, $\Phi_0 = 1$; compute then the SDE satisfied by (Z_t), using the integration by parts formula. Set next $Y_t = \log(Z_t)$ and search for the SDE satisfied by (Y_t), using Ito-Doeblin's formula (and without worrying about the fact that $\log(x)$ is not defined for $x \leq 0$: we are looking for a solution here, whatever it takes; formal verifications come after).

Exercise 2. (Brownian bridge)
Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. One considers the following linear SDE:

$$dX_t = -\frac{X_t}{1-t}\, dt + dB_t, \quad 0 < t < 1, \qquad X_0 = 0.$$

a) Find the solution $(X_t,\, t \in [0,1])$ of this equation. Notice that $X_1 = 0$.

b) Since the above equation is linear, (X_t) is a Gaussian process. Compute its mean and covariance.

Exercise 3. (Brownian motion on the circle)
Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Consider the following two-dimensional SDE:

$$\begin{cases} dX_t = -\dfrac{1}{2}\, X_t\, dt - Y_t\, dB_t, \quad t > 0, \qquad X_0 = 1, \\[2ex] dY_t = -\dfrac{1}{2}\, Y_t\, dt + X_t\, dB_t, \quad t > 0, \qquad Y_0 = 0. \end{cases}$$

a) Using (the generalized version of) Ito-Doeblin's formula, show that $X_t^2 + Y_t^2 = 1$ for all $t \in \mathbb{R}_+$.

b) Let

$$Z_t = \int_0^t X_s\, dY_s - \int_0^t Y_s\, dX_s, \quad t \in \mathbb{R}_+.$$

Compute Z_t.

c) By a), there exists a process $(\Theta_t,\, t \in \mathbb{R}_+)$ such that $X_t = \cos(\Theta_t)$ and $Y_t = \sin(\Theta_t)$. Using again Ito-Doeblin's formula, express (Z_t) in terms of (Θ_t). What can you deduce on the process (Θ_t)?