Homework 2

Exercise 1. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Let also $(X_t,\, t \in \mathbb{R}_+)$ and $(Y_t,\, t \in \mathbb{R}_+)$ be the processes defined as

$$X_t = \sin(B_t), \quad Y_t = e^{t/2} \sin(B_t).$$

a) Using Ito-Doeblin's formula, decide which of the above two processes is a martingale.

b) Compute the following quadratic variations and covariations:

$$\langle X \rangle_t, \quad \langle Y \rangle_t, \quad \langle X, B \rangle_t, \quad \langle Y, B \rangle_t, \quad \langle X, Y \rangle_t.$$

Exercise 2. Let $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be a filtration and $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Let $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ be two continuous functions, and let $(X_t,\, t \in \mathbb{R}_+)$ be a continuous process adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s \quad a.s. \quad \forall t \in \mathbb{R}_+.$$

We assume here that μ and σ are such that the process X is a semi-martingale as defined in class.

a) Using (the generalized version of) Ito-Doeblin's formula, show that if μ and σ satisfy

$$2x\, \mu(x) + \sigma(x)^2 = 0, \quad \forall x \in \mathbb{R},$$

then the process $(X_t^2,\, t \in \mathbb{R}_+)$ is a martingale (*NB:* leave aside the technical conditions!).

b) In the general case, compute the quadratic variation of both $(X_t,\, t \in \mathbb{R}_+)$ and $(X_t^2,\, t \in \mathbb{R}_+)$.

Exercise 3. Let $(X_t,\, t \in \mathbb{R}_+)$ and $(Y_t,\, t \in \mathbb{R}_+)$ be two continuous semi-martingales such that

$$\mathbb{E}\left(\int_0^t X_s^2\, d\langle Y \rangle_s\right) < \infty \quad \text{and} \quad \mathbb{E}\left(\int_0^t Y_s^2\, d\langle X \rangle_s\right) < \infty, \quad \forall t \in \mathbb{R}_+.$$

The generalized Fisk-Stratonovič integral is defined as

$$\int_0^t X_s \circ dY_s := \int_0^t X_s\, dY_s + \frac{1}{2}\, \langle X, Y \rangle_t, \quad t \in \mathbb{R}_+.$$

a) Rewrite the integration by parts formula using Fisk-Stratonovič integrals.

b) Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $X_t = \exp(B_t)$, $Y_t = B_t - t/2$. Show that the process $(M_t,\, t \in \mathbb{R}_+)$ defined as

$$M_t = \int_0^t X_s \circ dY_s, \quad t \in \mathbb{R}_+,$$

is a martingale.