Homework 11

Exercise 1. Let B be a standard (one-dimensional) Brownian motion and M be the martingale defined as

$$M_t = \int_0^t s\, dB_s, \quad t \in \mathbb{R}_+.$$

a) Compute the quadratic variation of M.

For $s \in \mathbb{R}_+$, let us define

$$\tau(s) = \inf\{t \geq 0 \ : \ \langle M \rangle_t \geq s\}.$$

b) Compute explicitly $\tau(s)$.

Let W be the process defined as

$$W_s = M(\tau(s)), \quad s \in \mathbb{R}_+.$$

c) Let $s_2 \geq s_1 \geq 0$. Compute $\mathbb{E}(W_{s_2} - W_{s_1})$ and $\mathbb{E}((W_{s_2} - W_{s_1})^2)$.

d) Do you have an idea of what type of process W could be?

Exercise 2. Let B be a standard (one-dimensional) Brownian motion and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as $f(x) = |x|$, $x \in \mathbb{R}$. Applying formally Ito-Doeblin's formula to $f(B_t)$, neglecting the fact that f is not twice continuously differentiable at $x = 0$ (not even once, actually), gives

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + 0 \quad a.s., \quad \forall t \in \mathbb{R}_+, \tag{1}$$

as

$$f'(x) = \operatorname{sgn}(x) = \begin{cases} +1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \end{cases}$$

and $f''(x) = 0$, for all $x \neq 0$.

a) Question: can the above formula possibly hold? Justify your answer.

Let us now define

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|B_s| < \varepsilon\}}\, ds.$$

L_t can be thought of as the "time spent in $x = 0$ by the Brownian motion over the period $[0, t]$".

b) Fact: L_t is typically non-zero. Is this fact surprising to you? Justify your opinion.

It turns out that L_t is the missing term in (1), that is, we actually have

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

c) Show that

$$\mathbb{E}(|B_t|) = \mathbb{E}\left(\int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t \right), \quad \forall t \in \mathbb{R}_+.$$

For the computation of $\mathbb{E}(L_t)$, you are allowed to permute limits and integrals without asking too many questions!