Homework 10

NB: In the three exercises below, we are dealing with functions which are not necessarily twice continuously differentiable in $\underline{x} = 0$. Nevertheless, Ito-Doeblin's formula still applies here, as we are considering multidimensional Brownian motions that do not come back too often to the position $\underline{x} = 0$.

Exercise 1. Let $\underline{B}$ be a standard 3-dimensional Brownian motion and let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined as $f(\underline{x}) = 1/\|\underline{x}\|$.

a) Show that $\Delta f(\underline{x}) = 0$ for all $\underline{x} \in \mathbb{R}^3$ such that $\underline{x} \neq 0$.

b) Let X be the process defined as $X_t = 1/\|\underline{B}_t\|$, $t \in \mathbb{R}_+$ (notice that $X_0 = \infty$ a.s., but we can live with that). What does part a) tell us about the process X?

c*) [optional] Show that for all $t > 0$, $\mathbb{E}(X_t) < \infty$.

d) Compute the quadratic variation of the process X. Is $\mathbb{E}(\langle X \rangle_t)$ finite?

e) Let $0 < a$ and $\tau_a = \inf\{t > 0 \,:\, \|\underline{B}_t\| \geq a\}$. Compute $\mathbb{E}(X_{\tau_a})$.

f) Given the two previous answers, do you think X is a martingale?

Exercise 2. Let $\underline{B}$ be a standard n-dimensional Brownian motion.

a) Show that the process M defined as $M_t = \|\underline{B}_t\|^2 - nt$, $t \in \mathbb{R}_+$, is a local martingale.

b) Explain why it is also a martingale.

Let also $a > 0$ and $\tau_a = \inf\{t > 0 \,:\, \|\underline{B}_t\| \geq a\}$.

c*) [optional] Explain why the optional stopping theorem can be applied to the martingale M with $\tau_1 = 0$ and $\tau_2 = \tau_a$, even though the stopping time τ_a is unbounded.

d) Deduce the value of $\mathbb{E}(\tau_a)$ (this should remember you something, by the way).

Exercise 3. Let $n \geq 2$ and $\underline{B}$ be a standard n-dimensional Brownian motion. Let X be the Bessel process defined as $X_t = \|\underline{B}_t\|$, $t \in \mathbb{R}_+$. X was shown in Homework 8, Exercise 1, to satisfy the SDE

$$dX_t = \frac{n-1}{2X_t}\, dt + dW_t, \quad X_0 = 0,$$

where W is a standard one-dimensional Brownian motion.

a) Let Y be the process defined as $Y_t = f(X_t)$, $t \in \mathbb{R}_+$, where $f : \mathbb{R} \to \mathbb{R}$ is some function such that $f(1) = 0$ and $f'(1) = \mathbf{1}$. What function f should we take in order for Y to be a local martingale?

b) With the above choice of f, is the process Y also a martingale? Justify your answer.