Homework 1

Exercise 0*. Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

a) Show that $\mathrm{Cov}(M_t, M_s)$ is only a function of $t \wedge s := \min(t,s)$, and not of t and s separately.

b) Show that $\mathrm{Var}(M_t) \geq \mathrm{Var}(M_s)$, if $t > s \geq 0$.

c) Let $(N_t,\, t \in \mathbb{R}_+)$ be another martingale with respect to the same filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, such that $\mathbb{E}(N_0) = \mathbb{E}(M_0)$ and $N_t \geq M_t$ a.s., for all $t \in \mathbb{R}_+$. Show that $N_t = M_t$ a.s., for all $t \in \mathbb{R}_+$.

Exercise 1. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and let

$$M_t = \int_0^t s\, dB_s, \quad N_t = \int_0^t s\, dM_s.$$

a) Compute the quadratic variation of M.

b) Compute the quadratic variation of N.

c) Knowing that the process N may be written as $N_t = \int_0^t f(s)\, dB_s$, can you deduce an expression for the function $f(s)$?

Exercise 2. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and let

$$M_t = \int_0^t e^s\, dB_s, \quad N_t = \int_0^t e^{-s}\, dM_s.$$

a) Compute the quadratic variation of M.

b) Compute the quadratic variation of N.

c) What can you deduce on the process N?

Exercise 3. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and let

$$M_t = \int_0^t e^s\, dB_s, \quad N_t = \int_0^t e^{-s}\, dB_s.$$

a) Compute the quadratic covariation of B and M.

b) Deduce from a) the value of $\mathbb{E}(B_t\, M_t)$.

c) Compute the quadratic covariation of M and N.

d) Deduce from c) the value of

$$\mathbb{E}\left(\int_0^t s\, dM_s \int_0^t s\, dN_s\right).$$