Solutions 9

1. a) In this example,

$$f^{(i)}(t,\underline{x}) = \mu_i\, x_i \quad \text{and} \quad g^{(i,j)}(t,\underline{x}) = \sigma_{ij}\, x_i.$$

For simplicity, we will drop the dependency on t in the following. The diffusion matrix G is given by

$$G^{(i,k)}(\underline{x}) = x_i\, x_k \sum_{j=1}^{2} \sigma_{ij}\, \sigma_{kj}$$

or in matrix form,

$$G(\underline{x}) = \begin{pmatrix} x_1^2\,(\sigma_{11}^2+\sigma_{12}^2) & x_1x_2\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}) \\ x_1x_2\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}) & x_2^2\,(\sigma_{21}^2+\sigma_{22}^2) \end{pmatrix}$$

The diffusion $\underline{X}$ is non-degenerate in D if and only if $\det G(\underline{x}) \neq 0$, for all $x \in D$. As

$$\det G(\underline{x}) = x_1^2x_2^2\,\left((\sigma_{11}^2+\sigma_{12}^2)\,(\sigma_{21}^2+\sigma_{22}^2) - (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})^2\right) = x_1^2x_2^2\,(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})^2,$$

we obtain that

$$\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21} \neq 0$$

is a necessary and sufficient condition for the diffusion $\underline{X}$ to be non-degenerate in D.

b) Let $\Delta = \sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21} \neq 0$. We can therefore invert the matrix G and obtain

$$\begin{aligned} G^{-1}(\underline{x}) &= \frac{1}{x_1^2x_2^2\,\Delta^2} \begin{pmatrix} x_2^2\,(\sigma_{21}^2+\sigma_{22}^2) & -x_1x_2\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}) \\ -x_1x_2\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}) & x_1^2\,(\sigma_{11}^2+\sigma_{12}^2) \end{pmatrix} \\ &= \frac{1}{\Delta^2} \begin{pmatrix} \dfrac{\sigma_{21}^2+\sigma_{22}^2}{x_1^2} & -\dfrac{\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}}{x_1\,x_2} \\ -\dfrac{\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}}{x_1\,x_2} & {\dfrac{\sigma_{11}^2+\sigma_{12}^2}{x_2}}^2 \end{pmatrix} \end{aligned}$$

Following the class, we then define

$$\begin{aligned} h^{(1)}(\underline{x}) &= f^{(1)}(\underline{x})\,(G^{-1})^{(1,1)}(\underline{x}) + f^{(2)}(\underline{x})\,(G^{-1})^{(2,1)}(\underline{x}) = \frac{\mu_1\,(\sigma_{21}^2+\sigma_{22}^2)}{x_1\,\Delta^2} - \frac{\mu_2\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})}{x_1\,\Delta^2}, \\ h^{(2)}(\underline{x}) &= f^{(1)}(\underline{x})\,(G^{-1})^{(1,2)}(\underline{x}) + f^{(2)}(\underline{x})\,(G^{-1})^{(2,2)}(\underline{x}) = -\frac{\mu_1\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})}{x_2\,\Delta^2} + \frac{\mu_2\,(\sigma_{11}^2+\sigma_{12}^2)}{x_2\,\Delta^2}, \end{aligned}$$

and

$$\begin{aligned} Z_t^{(1)} &= \int_0^t g^{(1,1)}(\underline{X}_s)\,dB_s^{(1)} + \int_0^t g^{(1,2)}(\underline{X}_s)\,dB_s^{(2)} = \sigma_{11}\int_0^t X_s^{(1)}\,dB_s^{(1)} + \sigma_{12}\int_0^t X_s^{(1)}\,dB_s^{(2)}, \\ Z_t^{(2)} &= \int_0^t g^{(2,1)}(\underline{X}_s)\,dB_s^{(1)} + \int_0^t g^{(2,2)}(\underline{X}_s)\,dB_s^{(2)} = \sigma_{21}\int_0^t X_s^{(2)}\,dB_s^{(1)} + \sigma_{22}\int_0^t X_s^{(2)}\,dB_s^{(2)}. \end{aligned}$$

Finally,

$$\begin{aligned}
M_t &= -\int_0^t h^{(1)}(\underline{\mathrm{X}}_s)\,dZ_s^{(1)} - \int_0^t h^{(2)}(\underline{\mathrm{X}}_s)\,dZ_s^{(2)} \\
&= -\frac{\mu_1\,(\sigma_{21}^2+\sigma_{22}^2) - \mu_2\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})}{\Delta^2}\left(\sigma_{11}\,B_t^{(1)} + \sigma_{12}\,B_t^{(2)}\right) \\
&\quad -\frac{\mu_2\,(\sigma_{11}^2+\sigma_{12}^2) - \mu_1\,(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})}{\Delta^2}\left(\sigma_{21}\,B_t^{(1)} + \sigma_{22}\,B_t^{(2)}\right). \\
&= -\frac{\mu_1\sigma_{22}(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})}{\Delta^2}\,B_t^{(1)} - \frac{\mu_2\sigma_{12}(\sigma_{12}\sigma_{21}-\sigma_{11}\sigma_{22})}{\Delta^2}\,B_t^{(1)} \\
&\quad -\frac{\mu_1\sigma_{21}(\sigma_{12}\sigma_{21}-\sigma_{11}\sigma_{22})}{\Delta^2}\,B_t^{(2)} - \frac{\mu_2\sigma_{11}(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})}{\Delta^2}\,B_t^{(2)} \\
&= -\frac{\mu_1\sigma_{22}-\mu_2\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\,B_t^{(1)} - \frac{\mu_2\sigma_{11}-\mu_1\sigma_{21}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\,B_t^{(2)}.
\end{aligned}$$

NB: For a diagonal matrix ($\sigma_{12}=\sigma_{21}=0$), this gives $M_t = -\dfrac{\mu_1}{\sigma_{11}}\,B_t^{(1)} - \dfrac{\mu_2}{\sigma_{22}}\,B_t^{(2)}$.

2. a) By Ito-Doeblin's formula, we have

$$u(\underline{\mathrm{B}}_t^{\mathrm{X}}) - u(\underline{\mathrm{B}}_0^{\mathrm{X}}) = \sum_{i=1}^n \int_0^t u'_{x_i}(\underline{\mathrm{B}}_s^{\mathrm{X}})\,dB_s^{(i)} + \frac{1}{2}\int_0^t \Delta u(\underline{\mathrm{B}}_s^{\mathrm{X}})\,ds.$$

Since $\frac{1}{2}\,\Delta u(\underline{\mathrm{x}}) = -g(\underline{\mathrm{x}})$, we deduce that $u(\underline{\mathrm{B}}_t^{\mathrm{X}}) + \int_0^t g(\underline{\mathrm{B}}_s^{\mathrm{X}})\,ds$ is a martingale. So the optional stopping theorem applied between times $\tau_1 = 0$ and $\tau_2 = \tau$ gives

$$u(\underline{\mathrm{x}}) = \mathbb{E}(u(\underline{\mathrm{B}}_0^x)) = \mathbb{E}\left(u(\underline{\mathrm{B}}_\tau^{\mathrm{X}}) + \int_0^\tau g(\underline{\mathrm{B}}_s^{\mathrm{X}})\,ds\right) = \mathbb{E}\left(\int_0^\tau g(\underline{\mathrm{B}}_s^{\mathrm{X}})\,ds\right),$$

as $u(\underline{\mathrm{B}}_\tau^{\mathrm{X}}) = 0$ (by the boundary condition of the PDE).

b) A direct computation gives $\Delta u(\underline{\mathrm{x}}) \equiv -2$, so the right choice is $g(\underline{\mathrm{x}}) \equiv 1$.

c) Applying the formula obtained in part (a), we get

$$u(\underline{\mathrm{x}}) = \mathbb{E}\left(\int_0^\tau 1\,ds\right) = \mathbb{E}(\tau),$$

so $\mathbb{E}(\tau) = \frac{1-\|\underline{\mathrm{x}}\|^2}{n}$.

d) For $\underline{\mathrm{x}} = 0$, we therefore obtain $\mathbb{E}(\tau) = \frac{1}{n}$, i.e., the expectation of the first exit time of the Brownian motion starting from the center of the unit ball decreases with the dimension n.

e) A possible interpretation of this is that the multi-dimensional Brownian motion has more "degrees of freedom" as the dimension increases, and thus moves faster out of the unit ball (the situation would certainly not be the same for the *cube* $[-1,1]^n$).