Solutions 8

1. a) Since W_t is a sum of stochastic integrals, it is a (continuous) martingale. Its quadratic variation is given by

$$\langle W\rangle_t = \sum_{i,k=1}^n \int_0^t \frac{B_s^{(i)}\, B_s^{(k)}}{\|\underline{B}_s\|^2}\, d\langle B^{(i)}, B^{(k)}\rangle_s = \sum_{i=1}^n \int_0^t \frac{(B_s^{(i)})^2}{\|\underline{B}_s\|^2}\, ds = \int_0^t 1\, ds = t.$$

Therefore, by Lévy's theorem, W_t is a standard Brownian motion.

b) For $\underline{x} \neq 0$, we have

$$f'_{x_i}(\underline{x}) = \frac{x_i}{\|\underline{x}\|} \quad \text{and} \quad f''_{x_i,x_k}(\underline{x}) = \frac{\delta_{ik}}{\|\underline{x}\|} - \frac{x_i\, x_k}{\|\underline{x}\|^3},$$

so $\Delta f(\underline{x}) = \dfrac{n-1}{\|\underline{x}\|}$ and

$$\|\underline{B}_t\| = \sum_{i=1}^n \int_0^t \frac{B_s^{(i)}}{\|\underline{B}_s\|}\, dB_s^{(i)} + \frac{1}{2}\int_0^t \frac{n-1}{\|\underline{B}_s\|}\, ds.$$

By using the result in part a), we therefore obtain that $R_t = \|\underline{B}_t\|$ satisfies the equation

$$R_t = W_t + \int_0^t \frac{n-1}{2R_s}\, ds.$$

2. a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$ and $\underline{z}_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$.

b) By the class, we know that $\Phi_t = \exp(tA)$, where

$$\exp(tA) = \sum_{n=0}^\infty \frac{t^n A^n}{n!}.$$

As $A^2 = -I$, we obtain that $A^{2k} = (-1)^k\, I$, $A^{2k+1} = (-1)^k\, A$ and

$$\exp(tA) = \sum_{k=0}^\infty (-1)^k\, \frac{t^{2k}}{(2k)!}\, I + \sum_{k=0}^\infty (-1)^k\, \frac{t^{2k+1}}{(2k+1)!}\, A = \cos(t)\, I + \sin(t)\, A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

c) These formulas are easily proved using trigonometric relations such as $\cos(-t) = \cos(t)$ and $\cos(t+s) = \cos(t)\cos(s) - \sin(t)\sin(s)$.

d) By the class, we know that

$$\underline{Z}_t = \int_0^t \Phi_{t-s}\, \Sigma\, d\underline{B}_s + \Phi_t\, \underline{z}_0.$$

Therefore,

$$X_t = \sigma \int_0^t \sin(t-s)\, dB_s + \cos(t)\, x_0 \quad \text{and} \quad Y_t = \sigma \int_0^t \cos(t-s)\, dB_s - \sin(t)\, x_0.$$

e) $\dfrac{dX_t}{dt} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(X_{t+\varepsilon} - X_t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \sin(t+\varepsilon-s)\, dB_s + Y_t = Y_t$,
as $\mathbb{E}((\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \sin(t+\varepsilon-s)\, dB_s)^2) = \frac{1}{\varepsilon^2}\int_t^{t+\varepsilon} \sin(t+\varepsilon-s)^2\, ds = \frac{1}{2\varepsilon^2}(\varepsilon - \sin(2\varepsilon)/2) = O(\varepsilon)$.