Solutions 7

1. a) Ito-Doeblin's formula applied to $v(X_t)$ gives

$$\begin{aligned} v(X_t) - v(x_0) &= \int_0^t v'(X_s)\, dX_s + \frac{1}{2}\int_0^t v''(X_s)\, d\langle X\rangle_s \\ &= \int_0^t v'(X_s)\, f(X_s)\, ds + \int_0^t v'(X_s)\, g(X_s)\, dB_s + \frac{1}{2}\int_0^t v''(X_s)\, g(X_s)^2\, ds \\ &= \int_0^t Av(X_s)\, ds + \int_0^t v'(X_s)\, g(X_s)\, dB_s, \end{aligned}$$

therefore the result.

b) From a), we deduce that

$$\lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}(v(X_t) - v(x_0)) = \lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}\left(\int_0^t Av(X_s)\, ds\right) = Av(x_0)$$

by the fact that Av is a continuous function and the mean value theorem.

c) Setting $v(x) = x$ in the previous equality, we obtain

$$\lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}(X_t - x_0) = 1 \cdot f(x_0) + 0 \cdot \frac{1}{2}\, g(x_0)^2 = f(x_0).$$

4) Noticing that $(X_t - x_0)^2 = (X_t^2 - x_0^2) - 2x_0\,(X_t - x_0)$, and applying b) with $v(x) = x^2$ and $v(x) = x$, we obtain

$$\lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}((X_t - x_0)^2) = 2x_0 \cdot f(x_0) + 2 \cdot \frac{1}{2}\, g(x_0)^2 - 2x_0 \cdot f(x_0) = g(x_0)^2.$$

2. a) By the class, we know that there exists a probability measure $\widetilde{\mathbb{P}}_T$ under which $(X_t,\, t \in [0,T])$ is a Brownian motion starting at point x_0 at time $t = 0$. We also know that the premium of the option at time $t < T$ when $X_t = x$ is given by

$$c(t,x) = \widetilde{\mathbb{E}}_T(\max(X_T^{t,x} - K, 0)).$$

where $(X_s^{t,x},\, s \in [t,T])$ is a Brownian motion starting at point x at time t. Therefore,

$$\begin{aligned} c(t,x) &= \int_{\mathbb{R}} \max(y - K, 0)\, p_{T-t}(x-y)\, dy = \int_{\mathbb{R}} \max(x - y - K, 0)\, p_{T-t}(y)\, dy \\ &= \int_{-\infty}^{x-K} (x - y - K)\, p_{T-t}(y)\, dy \end{aligned}$$

where p_{T-t} if the pdf of the $\mathcal{N}(0, T-t)$ distribution. After integration, this can be further written as

$$c(t,x) = (x-K)\, N\left(\frac{x-K}{\sqrt{T-t}}\right) + \sqrt{\frac{T-t}{2\pi}}\, \exp\left(-\frac{(x-K)^2}{2(T-t)}\right).$$

Remark: Notice also that the premium $c(t,x)$ satisfies the PDE

$$c'_t(t,x) + \frac{1}{2}\, c''_{xx}(t,x) = 0, \quad c(T,x) = \max(x-K,0).$$

b) By the class, the hedging strategy $(\phi_t,\, t \in [0,T])$ is given by $\phi_t = c'_x(t,X_t)$. Rather than deriving the last formula obtained for $c(t,x)$, let us move one step back to obtain

$$c'_x(t,x) = \frac{\partial}{\partial x} \int_{-\infty}^{x-K} (x-y-K)\, p_{T-t}(y)\, dy = \int_{-\infty}^{x-K} p_{T-t}(y)\, dy = N\left(\frac{x-K}{\sqrt{T-t}}\right).$$

Reminder:

$$\frac{\partial}{\partial x} \int_{-\infty}^{a(x)} f(x,y)\, dy = f(x,a(x))\, a'(x) + \int_{-\infty}^{a(x)} f'_x(x,y)\, dy,$$

and notice that the first term is zero in the computation of $c'_x(t,x)$.

3. By the class, the hedging strategy $(\phi_t,\, t \in [0,T])$ is given by $\phi_t = c'_x(t,X_t)$, where $c(t,x)$ is the premium at time $t < T$ when $X_t = x$. Remember also that in the Black-Scholes model with time-independent coefficients, $c(t,x)$ is given by

$$c(t,x) = \int_{y(t,x)}^{\infty} dy\, p_{T-t}(y) \left(x\, e^{\sigma y - \frac{\sigma^2 (T-t)}{2}} - K\right),$$

where

$$y(t,x) = \frac{1}{\sigma}\left(\log\left(\frac{K}{x}\right) + \frac{\sigma^2\,(T-t)}{2}\right)$$

is defined such that the expression in parentheses in the above formula is equal to zero when we set $y = y(t,x)$. Using again the above reminder, we obtain

$$\begin{aligned} c'_x(t,x) \quad &= \quad -p_{T-t}(y(t,x)) \left(x\, e^{\sigma y(t,x) - \frac{\sigma^2 (T-t)}{2}} - K\right) y'_x(t,x) \\ &\quad + \int_{y(t,x)}^{\infty} dy\, \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-\frac{y^2}{2(T-t)}}\, e^{\sigma y - \frac{\sigma^2 (T-t)}{2}}. \end{aligned}$$

By the remark just made about $y(t,x)$, the first term on the right-hand side is zero, so

$$c'_x(t,x) = \int_{y(t,x)}^{\infty} dy\, \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-\frac{(y-\sigma(T-t))^2}{2(T-t)}} = \ldots = N(d_1(t,x)),$$

where

$$d_1(t,x) = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{x}{K}\right) + \frac{\sigma^2 (T-t)}{2}\right)$$

Remark: From the formula

$$c(t,x) = x\, N(d_1) - K\, N(d_2),$$

one could be tempted to deduce that "of course"

$$c'_x(t,x) = N(d_1).$$

Notice nevertheless that both d_1 and d_2 depend on x, i.e., there is some cancellation going on here!