Solutions 6

1. a) By the integration by parts formula applied to $M_t = C_t\, u(t, B_t^{t_0,x_0})$, we have

$$dM_t = (dC_t)\, u(t, B_t^{t_0,x_0}) + C_t\, du(t, B_t^{t_0,x_0}) + 0.$$

But

$$dC_t = c(t, B_t^{t_0,x_0})\, C_t\, dt$$

so by Ito-Doeblin's formula, we obtain

$$du(t, B_t^{t_0,x_0}) = u_t'(t, B_t^{t_0,x_0})\, dt + u_x'(t, B_t^{t_0,x_0})\, dB_t + \frac{1}{2} u_{xx}''(t, B_t^{t_0,x_0})\, dt.$$

Therefore,

$$\begin{aligned} dM_t &= C_t \left(u_t'(t, B_t^{t_0,x_0}) + \frac{1}{2}\, u_{xx}''(t, B_t^{t_0,x_0}) + c(t, B_t^{t_0,x_0})\, u(t, B_t^{t_0,x_0}) \right) dt + C_t\, u_x'(t, B_t^{t_0,x_0})\, dB_t \\ &= C_t\, u_x'(t, B_t^{t_0,x_0})\, dB_t, \end{aligned}$$

since u satisfies the equation mentioned in the problem set. We deduce that M is a martingale.

b) We have

$$\begin{aligned} u(t_0, x_0) &= \mathbb{E}(u(t_0, B_{t_0}^{t_0,x_0})) = \mathbb{E}(M_{t_0}) = \mathbb{E}(M_T) \qquad \text{(notice that } C_{t_0} = 1) \\ &= \mathbb{E}(C_T\, u(T, B_T^{t_0,x_0})) = \mathbb{E}\left(\exp\left(\int_{t_0}^T c(s, B_s^{t_0,x_0})\, ds \right)\, h(B_T^{t_0,x_0}) \right). \end{aligned}$$

2. a) Here $f(t,x) = -ax$ and $g(t,x) = 1$. Following therefore what has been done in class, let $T > 0$ be fixed and let us define the process $(X_t^x,\, t \in [0,T])$ as being the solution of

$$dX_t^x = -aX_t^x\, dt + dB_t, \quad X_0 = x.$$

The solution of this SDE is an Ornstein-Uhlenbeck process starting at $x \in \mathbb{R}$:

$$X_t^x = e^{-at}\, x + \int_0^t e^{-a(t-s)}\, dB_s,$$

and $u(T,x) = \mathbb{E}(u_0(X_T^x))$.

b) We know that X_T^x is a $\mathcal{N}\left(e^{-aT}\, x, \sigma_T^2\right)$ random variable, where

$$\sigma_T^2 = \int_0^T e^{-2a(T-s)}\, ds = \frac{1 - e^{-2aT}}{2a}.$$

So

$$u(T,x) = \mathbb{E}(u_0(X_T^x)) = \int_{\mathbb{R}} dy\, u_0(y)\, \frac{1}{\sqrt{2\pi\sigma_T^2}}\, \exp\left(-\frac{\left(y - e^{-aT}\, x\right)^2}{2\sigma_T^2} \right).$$

For the particular case where $u_0(y) = \max(y, 0)$, we can compute this further:

$$\begin{aligned}
u(T,x) &= \int_0^\infty dy\, y \frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left(-\frac{\left(y - e^{-aT}\,x\right)^2}{2\sigma_T^2}\right) \\
&= \int_{-e^{-aT}\,x}^\infty dz\,(z + e^{-aT}\,x)\,\frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left(-\frac{z^2}{2\sigma_T^2}\right) \\
&= -\frac{\sigma_T^2}{\sqrt{2\pi\sigma_T^2}} \exp\left(-\frac{z^2}{2\sigma_T^2}\right)\Bigg|_{-e^{-aT}\,x}^{\infty} + e^{-aT}\,x\,\left(1 - N\left(-\frac{e^{-aT}\,x}{\sigma_T}\right)\right) \\
&= \frac{\sigma_T}{\sqrt{2\pi}} \exp\left(-\frac{e^{-2aT}\,x^2}{2\sigma_T^2}\right) + e^{-aT}\,x\,N\left(\frac{e^{-aT}\,x}{\sigma_T}\right).
\end{aligned}$$

3. a) Here $f(t,x) = -x/t$ and $g(t,x) = 1$. Following therefore what has been done in class, let $T > 0$ be fixed and let us define the process $(X_t^x,\, t \in [0,T])$ as being the solution of

$$dX_t^x = -\frac{X_t^x}{T-t}\,dt + dB_t, \quad X_0 = x.$$

The solution of this SDE is a Brownian bridge between 0 and T, starting at $x \in \mathbb{R}$:

$$X_t^x = \frac{T-t}{T}\,x + \int_0^t \frac{T-t}{T-s}\,dB_s,$$

and $u(T,x) = \mathbb{E}(u_0(X_T^x))$???

b) The problem is: $X_T^x = 0$ a.s., for all $x \in \mathbb{R}$! So $u(T,x) = u_0(0)$ for all $T > 0$ and $x \in \mathbb{R}$??? This cannot be. The problem we have here is that the function $f(t,x) = -x/t$ is not Lipshitz on any time interval $[0,T]$, as the Lipschitz constant of the function is unbounded on such an interval. The existence (and uniqueness) of the solution of the PDE is therefore in question, and our method for obtaining a probabilistic representation of the solution certainly does not work here.