Solutions 5

1. a) Given the assumnptions made, we obtain that both $f(t,x) = \mu(t)\, x$ and $g(t,x) = \sigma(t)\, x$ are jointly continuous in (t,x) and also Lipshitz in x. Indeed, we have e.g. for f:

$$|f(t,x) - f(t,y)| \le |\mu(t)|\, |x-y| \le K_1\, |x-y|, \quad \forall t,x,y.$$

By the theorem seen in class, we therefore know that there exists a unique strong solution to the SDE. Let now $Y_t = \log(X_t)$. By Ito-Doeblin's formula,

$$dY_t = \frac{1}{X_t}\, dX_t - \frac{1}{2}\,\frac{1}{X_t^2}\, d\langle X\rangle_t.$$

As $d\langle X\rangle_t = \sigma(t)^2\, X_t^2\, dt$, we obtain that

$$dY_t = \mu(t)\, dt + \sigma(t)\, dB_t - \frac{1}{2}\,\sigma(t)^2\, dt,$$

i.e.

$$Y_t = y_0 + \int_0^t \left(\mu(s) - \frac{\sigma(s)^2}{2}\right) ds + \int_0^t \sigma(s)\, dB_s.$$

Therefore,

$$X(t) = x_0\, \exp\left(\int_0^t \left(\mu(s) - \frac{\sigma(s)^2}{2}\right) ds + \int_0^t \sigma(s)\, dB_s\right).$$

b) Let $M_t = -\int_0^t \frac{f(s,X_s)}{g(s,X_s)}\, dB_s = -\int_0^t \frac{\mu(s)}{\sigma(s)}\, dB_s$. We check that

$$\langle M\rangle_t = \int_0^t \frac{\mu(s)^2}{\sigma(s)^2}\, ds \le \frac{K_1^2}{K_2^2}\, t$$

The process $Y_t = \exp(M_t - \langle M\rangle_t/2)$ is therefore a martingale and $\widetilde{\mathbb{P}}_T(A) = \mathbb{E}(1_A\, Y_T)$ is a probability measure. By Girsanov's theorem, we know that $(X_t, t \in [0,T])$ is a martingale under $\widetilde{\mathbb{P}}_T$. In addition, the process defined as

$$\widetilde{B}_t = B_t - \langle M, B\rangle_t = B_t + \int_0^t \frac{\mu(s)}{\sigma(s)}\, ds$$

is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$, and

$$dX_t = \sigma(t)\, X_t\, d\widetilde{B}_t, \quad X_0 = 0,$$

so

$$X_t = x_0\, \exp\left(\int_0^t \sigma(s)\, d\widetilde{B}_s - \int_0^t \frac{\sigma(s)^2}{2}\, ds\right).$$

For ease of notation, let us write $\sigma_t^2 = \int_0^t \sigma(s)^2\, ds$. Notice also that under $\widetilde{\mathbb{P}}_T$, $\int_0^t \sigma(s)\, d\widetilde{B}_s$ ia a Gaussian random variable with zero mean and variance σ_t^2.

c) Following what has been done in class, we obtain that the premium is

$$\begin{aligned} c_0 &= \widetilde{\mathbb{E}}_T(\max(X_T - K, 0)) = \widetilde{\mathbb{E}}_T\left(\max\left(x_0 \exp\left(\int_0^T \sigma(s)\, d\widetilde{B}_s - \frac{\sigma_T^2}{2}\right) - K, 0\right)\right) \\ &= \int_{\mathbb{R}} dy\, p_T(y)\, \max\left(x_0 \exp\left(y - \frac{\sigma_T^2}{2}\right) - K, 0\right), \end{aligned}$$

where

$$p_T(y) = \frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left(-\frac{y^2}{2\,\sigma_T^2}\right).$$

Let us push the computation further and define $y_0 = \log(K/x_0) + \sigma_T^2/2$. We obtain that

$$\begin{aligned} c_0 &= x_0 \int_{y_0}^{\infty} dy\, \frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left(-\frac{(y-\sigma_T^2)^2}{2\sigma_T^2}\right) - K \int_{y_0}^{\infty} dy\, \frac{1}{\sqrt{2\pi\sigma_T^2}} \exp\left(-\frac{y^2}{2\,\sigma_T^2}\right) \\ &= x_0 \left(1 - N\left(\frac{y_0 - \sigma_T^2}{\sigma_T}\right)\right) - K\left(1 - N\left(\frac{y_0}{\sigma_T}\right)\right) = x_0\, N(d_1) - K\, N(d_2), \end{aligned}$$

where

$$d_1 = \frac{\sigma_T^2 - y_0}{\sigma_T} = \frac{1}{\sigma_T}\left(\log\left(\frac{x_0}{K}\right) + \frac{\sigma_T^2}{2}\right) \quad \text{and} \quad d_2 = -\frac{y_0}{\sigma_T} = \frac{1}{\sigma_T}\left(\log\left(\frac{x_0}{K}\right) - \frac{\sigma_T^2}{2}\right).$$

2. The premium C_t at time t must allow the seller to hedge the option with a strategy $(\phi_s, s \in [t, T])$ so as to recover the wealth $C_T = \max(X_T - K, 0)$ at time T. Therefore,

$$C_T = C_t + \int_t^T \phi_s\, dX_s = C_t + G_T - G_t,$$

where

$$G_t = \int_0^t \phi_s\, dX_s.$$

We have seen in class that under the probability measure $\widetilde{\mathbb{P}}_T$, X is a martingale, and so is the process G. Therefore,

$$\widetilde{\mathbb{E}}_T(C_T|\mathcal{F}_t) = \widetilde{\mathbb{E}}_T(C_t|\mathcal{F}_t) + \widetilde{\mathbb{E}}_T(G_T - G_t|\mathcal{F}_t) = C_t + \widetilde{\mathbb{E}}_T(G_T|\mathcal{F}_t) - G_t = C_t,$$

i.e.

$$C_t = \widetilde{\mathbb{E}}_T(C_T|\mathcal{F}_t) = \widetilde{\mathbb{E}}_T(\max(X_T - K, 0)|\mathcal{F}_t)$$

Besides, $dX_t = \sigma\, X_t\, d\widetilde{B}_t$, so

$$X_t = x_0\, e^{\sigma \widetilde{B}_t - \frac{\sigma^2 t}{2}}, \quad X_T = x_0\, e^{\sigma \widetilde{B}_T - \frac{\sigma^2 T}{2}} \quad \text{and} \quad X_T = X_t\, e^{\sigma\,(\widetilde{B}_T - \widetilde{B}_t) - \frac{\sigma^2\,(T-t)}{2}}.$$

Given that X_t is $\mathcal{F}_t$-measurable and $\widetilde{B}_T - \widetilde{B_t}$ is independent of $\mathcal{F}_t$, we obtain that

$$C_t = \widetilde{\mathbb{E}}_T(\max(X_T - K, 0)|\mathcal{F}_t) = c(T - t, X_t),$$

where

$$
\begin{aligned}
c(T-t,x) &= \widetilde{\mathbb{E}}_T\left(\max\left(x\,\exp\left(\sigma(\widetilde{B}_T-\widetilde{B}_t)-\frac{\sigma^2}{2}\,(T-t)\right)-K,0\right)\right)\\
&= \int_{\mathbb{R}} dy\, p_{T-t}(y)\, \max\left(x\,\exp\left(\sigma y-\frac{\sigma^2}{2}\,(T-t)\right)-K,0\right)
\end{aligned}
$$

and

$$
p_{T-t}(y) = \frac{1}{\sqrt{2\pi(T-t)}}\,\exp\left(-\frac{y^2}{2(T-t)}\right).
$$

Defining now $y_0 = \frac{1}{\sigma}(\log(K/x)+\sigma^2\,(T-t)/2)$, we obtain that

$$
\begin{aligned}
c(T-t,x) &= x\int_{y_0}^{\infty} dy\,\frac{1}{\sqrt{2\pi(T-t)}}\,\exp\left(-\frac{(y-\sigma\,(T-t))^2}{2\,(T-t)}\right)\\
&\quad -K\int_{y_0}^{\infty} dy\,\frac{1}{\sqrt{2\pi(T-t)}}\,\exp\left(-\frac{y^2}{2\,(T-t)}\right)\\
&= x\left(1-N\left(\frac{y_0-\sigma\,(T-t)}{\sqrt{T-t}}\right)\right)-K\left(1-N\left(\frac{y_0}{\sqrt{T-t}}\right)\right)\\
&= x\,N(d_1)-K\,N(d_2),
\end{aligned}
$$

where

$$
d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{x}{K}\right)+\frac{\sigma^2\,(T-t)}{2}\right) \quad\text{and}\quad d_2 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{x}{K}\right)-\frac{\sigma^2\,(T-t)}{2}\right).
$$

Notice that this formula is actually the same as the formula seen in class with x_0 replaced by x, the value of the stock at time t, and T replaced by $T-t$, the time-to-maturity. We could therefore have guessed this formula for $c(T-t,x)$ from the start. Notice also that we now have a description of the option price as a process $C_t = c(T-t,X_t)$ evolving in time.

3. Following again what has been done in class, the premium c_0 is given by

$$
c_0 = \widetilde{\mathbb{E}}_T(1_{X_T>K}) = \widetilde{\mathbb{P}}_T(X_T>K).
$$

As $X_T = x_0\,\exp(\sigma\,\widetilde{B}_T-\sigma^2T/2)$ and $\widetilde{B}_T\sim\mathcal{N}(0,T)$ under $\widetilde{\mathbb{P}}_T$, we obtain that

$$
c_0 = \widetilde{\mathbb{P}}_T\left(x_0\,\exp\left(\sigma\,\widetilde{B}_T-\frac{\sigma^2 T}{2}\right)>K\right) = \int_{y_0}^{\infty} dy\,p_T(y),
$$

where

$$
y_0 = \frac{1}{\sigma}\left(\log\left(\frac{K}{x_0}\right)+\frac{\sigma^2\,T}{2}\right) \quad\text{and}\quad p_T(y) = \frac{1}{\sqrt{2\pi T}}\,\exp\left(-\frac{y^2}{2T}\right).
$$

So

$$
c_0 = 1-N\left(\frac{y_0}{\sqrt{T}}\right) = N\left(-\frac{y_0}{\sqrt{T}}\right) = N(d_2) \quad\text{where}\quad d_2 = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{x_0}{K}\right)-\frac{\sigma^2\,T}{2}\right).
$$

Notice that $c_0 = N(d_2)$ is a probability between 0 and 1 (which is consistent with the fact that the option payoff takes values in $\{0,1\}$). However, it is *not* the true probability that $X_T>K$ given that $X_0 = x_0$; it is actually this probability under the risk-neutral measure $\widetilde{\mathbb{P}}_T$, as the first line shows.