Solutions 4

1. a) Since X and Z are independent, $\mathbb{E}(Y|X) = \varphi(X)$, where

$$\varphi(x) = \mathbb{E}(\exp(-xZ - x^2/2)) = \exp(x^2/2 - x^2/2) = 1.$$

So $\mathbb{E}(Y|X) = 1$.

b) $\widetilde{\mathbb{P}}(\Omega) = \mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(1) = 1$.

c) By definition, we have

$$\widetilde{\mathbb{E}}(g(X+Z)) = \mathbb{E}(g(X+Z)\,\exp(-XZ-X^2/2)) = \mathbb{E}(\mathbb{E}(g(X+Z)\,\exp(-XZ-X^2/2)\,|\,X)) = \mathbb{E}(\psi(X)),$$

where

$$\begin{aligned}\psi(x) &= \mathbb{E}(g(x+Z)\,\exp(-xZ-x^2/2)) = \int_{\mathbb{R}} dz\, g(x+z)\,\exp(-xz-x^2/2)\,\frac{1}{\sqrt{2\pi}}\,\exp(-z^2/2)\\ &= \int_{\mathbb{R}} dz\, g(x+z)\,\frac{1}{\sqrt{2\pi}}\,\exp(-(x+z)^2/2) = \int_{\mathbb{R}} dy\, g(y)\,\frac{1}{\sqrt{2\pi}}\,\exp(-y^2/2),\end{aligned}$$

by the change of variable $y = x + z$. This function does not actually depend on x, so

$$\widetilde{\mathbb{E}}(g(X+Z)) = \mathbb{E}(\psi(X)) = \int_{\mathbb{R}} dy\, g(y)\,\frac{1}{\sqrt{2\pi}}\,\exp(-y^2/2),$$

which concludes the proof.

2. By the class, we know that if M is a martingale under $\mathbb{P}$ such that $\langle M\rangle_t \le Kt$ for all $t \in [0,T]$, then $(\widetilde{B}_t := B_t - \langle M, B\rangle_t,\ t \in [0,T])$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$, where $\widetilde{\mathbb{P}}_T$ is defined as

$$\widetilde{\mathbb{P}}_T(A) = \mathbb{E}(1_A\, Y_T) \quad\text{and}\quad Y_t = \exp(M_t - \langle M\rangle_t/2).$$

If we therefore set $M_t = \int_0^t g'(s)\,dB_s$, then we indeed have

$$\langle M\rangle_t = \int_0^t g'(s)^2\,ds \le \left(\sup_{s\in[0,T]} g'(s)^2\right) t = Kt$$

where $K = \sup_{s\in[0,T]} g'(s)^2 < \infty$ by assumption, and

$$\langle M, B\rangle_t = \int_0^t g'(s)\,ds = g(t) - g(0) = g(t),$$

so that $(\widetilde{B}_t = B_t - g(t))$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$. Therefore, for all $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0\le t\le T} |B_t - g(t)| \le \varepsilon\right) = \mathbb{P}\left(\sup_{0\le t\le T} |\widetilde{B}_t| \le \varepsilon\right) > 0,$$

since we know that

$$\widetilde{\mathbb{P}}_T \left(\sup_{0 \le t \le T} |\widetilde{B}_t| \le \varepsilon \right) > 0$$

and we also know that the two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}_T$ are equivalent, which means that $\mathbb{P}(A) > 0$ whenever $\widetilde{\mathbb{P}}_T(A) > 0$, for any $A \in \mathcal{F}$.

3. Following the class, the martingale M should be

$$M_t = -\int_0^t \left(-\frac{X_s}{1-s} \right) dB_s = \int_0^t \frac{X_s}{1-s} \, dB_s$$

If it was the case that for some constant $K > 0$, $\langle M \rangle_t \le Kt$ for all $t \in [0,1]$, then this would imply that

$$\mathbb{E}(\langle M \rangle_t) \le Kt, \quad \forall t \in [0,1].$$

We show below that such a constant does not exist. Indeed,

$$\mathbb{E}(\langle M \rangle_t) = \mathbb{E}(M_t^2) = \mathbb{E}\left(\int_0^t \frac{X_s^2}{(1-s)^2} \, ds \right) = \int_0^t \frac{\mathbb{E}(X_s^2)}{(1-s)^2} \, ds,$$

where (see Homework 3, Exercise 2)

$$X_s = \int_0^s \frac{1-s}{1-r} \, dB_r \quad \text{and} \quad \mathbb{E}(X_s^2) = \int_0^s \frac{(1-s)^2}{(1-r)^2} \, dr = s\,(1-s).$$

So

$$\mathbb{E}(\langle M \rangle_t) = \int_0^t \frac{s}{1-s} \, ds = t - \int_0^t \frac{1}{1-s} \, ds = t - \log(1-t) = t + \log\left(\frac{1}{1-t} \right).$$

This function tends to infinity as t gets close to 1, so there is indeed no constant $K > 0$ such that $\mathbb{E}(\langle M \rangle_t) \le Kt$ for all $t \in [0,1]$.

The reason why there cannot be a probability measure $\widetilde{\mathbb{P}}_1$ under which X is a martingale is that the value of X at $t = 1$ is deterministic, i.e. $\mathbb{P}(X_1 = 0) = 1$. This implies that for any equivalent probability measure $\widetilde{\mathbb{P}}_1$, we will also have $\widetilde{\mathbb{P}}_1(X_1 = 0) = 1$. But a process whose value at some time instant in the future is known with probability one cannot be a martingale, as this would imply that

$$X_t = \widetilde{\mathbb{E}}_1(X_1 | \mathcal{F}_t) = \widetilde{\mathbb{E}}_1(0 | \mathcal{F}_t) = 0, \quad \forall t \in [0,1],$$

which is obviously not the case here.