**Solutions 3**

**1.** a) Let $\Phi_t$ be the solution of $d\Phi_t = \mu\,\Phi_t\,dt$, $\Phi_0 = 1$: $\Phi_t$ is given by the expression $\Phi_t = \exp(\mu\,t)$.

b) Set $X_t = \Phi_t\,Z_t$; we have

$$\begin{aligned} dX_t &= Z_t\,d\Phi_t + \Phi_t\,dZ_t + 0 = \mu\,\Phi_t\,Z_t\,dt + \Phi_t\,dZ_t \\ &= \mu\,X_t\,dt + \sigma\,X_t\,dB_t, \end{aligned}$$

where the first line follows from the integration by parts formula and the second line is obtained by rewriting simply the equation for $X_t$. From the above equation, we obtain that $dZ_t = \sigma\,Z_t\,dB_t$. Moreover, $Z_0 = x_0$ and

$$\langle Z\rangle_t = \sigma^2 \int_0^t Z_s^2\,ds.$$

c) Set $Y_t = \log(Z_t)$; by Ito-Doeblin's formula, we have (reminder: $\log(x)' = \frac{1}{x}$ and $\log(x)'' = -\frac{1}{x^2}$):

$$Y_t - \log(x_0) = \int_0^t \frac{1}{Z_s}\,dZ_s - \frac{1}{2}\int_0^t \frac{1}{Z_s^2}\,d\langle Z\rangle_s = \sigma \int_0^t dB_s - \frac{\sigma^2}{2}\int_0^t ds = \sigma\,B_t - \frac{\sigma^2}{2}\,t,$$

therefore

$$X_t = \Phi_t\,\exp(Y_t) = x_0\,\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\,B_t\right).$$

**2.** a) By the class, the solution of this SDE reads $X_t = \displaystyle\int_0^t \frac{\Phi_t}{\Phi_s}\,dB_s$, where $d\Phi_t = -\dfrac{\Phi_t}{1-t}\,dt$, $\Phi_0 = 1$. Solving this ODE gives

$$\frac{\Phi_t'}{\Phi_t} = -\frac{1}{1-t} \quad\Rightarrow\quad \log(\Phi_t) = \log(1-t) + C \quad\Rightarrow\quad \Phi_t = 1-t \quad \text{since} \quad \Phi_0 = 1.$$

So $X_t = \displaystyle\int_0^t \frac{1-t}{1-s}\,dB_s$ and $X_1 = \displaystyle\int_0^1 0\,dB_s = 0$.

b) $\mathbb{E}(X_t) = 0$ and for $t \geq s$, we have

$$\mathrm{Cov}(X_t, X_s) = \mathbb{E}(X_t\,X_s) = (1-t)\,(1-s)\int_0^s \frac{1}{(1-r)^2}\,dr = (1-t)\,(1-s)\left(\frac{1}{1-s} - 1\right) = (1-t)\,s,$$

thus, for general $t, s$, we have

$$\mathrm{Cov}(X_t, X_s) = (1 - (t \vee s))\,(t \wedge s) \quad \text{et} \quad \mathrm{Var}(X_t) = t\,(1-t),$$

where $t \vee s := \max(t, s)$. Notice that for $t, s$ close to 0, we have

- $\mathrm{Cov}(X_t, X_s) \simeq t \wedge s$ (that is, $(X_t)$ resembles to a Brownian motion, close to $t = 0$);
- $\mathrm{Cov}(X_{1-t}, X_{1-s}) \simeq t \wedge s$ (that is, $(X_{1-t})$ also resembles to a Brownian motion, close to $t = 0$).

**3.** a) By Ito-Doeblin's formula, we have

$$\begin{aligned} X_t^2 &= 1+2\int_0^t X_s\,dX_s + \langle X\rangle_t = 1-\int_0^t X_s^2\,ds - 2\int_0^t X_s\,Y_s\,dB_s + \int_0^t Y_s^2\,ds, \\ Y_t^2 &= 0+2\int_0^t Y_s\,dY_s + \langle Y\rangle_t = -\int_0^t Y_s^2\,ds + 2\int_0^t Y_s\,X_s\,dB_s + \int_0^t X_s^2\,ds, \end{aligned}$$

so $X_t^2 + Y_t^2 = 1$.

b) We have

$$Z_t = -\frac{1}{2}\int_0^t X_s\,Y_s\,ds + \int_0^t X_s^2\,dB_s + \frac{1}{2}\int_0^t Y_s\,X_s\,ds + \int_0^t Y_s^2\,dB_s = 0 + \int_0^t (X_s^2+Y_s^2)\,dB_S = \int_0^t 1\,dB_s = B_t.$$

c) We have

$$\begin{aligned} dX_t &= d\cos(\Theta_t) = -\sin(\Theta_t)\,d\Theta_t - \frac{1}{2}\cos(\Theta_t)\,d\langle\Theta\rangle_t, \\ dY_t &= d\sin(\Theta_t) = +\cos(\Theta_t)\,d\Theta_t - \frac{1}{2}\sin(\Theta_t)\,d\langle\Theta\rangle_t. \end{aligned}$$

Therefore, by a computation very close to the one above,

$$Z_t = \int_0^t \cos(\Theta_s)\,d\sin(\Theta_s) - \int_0^t \sin(\Theta_s)\,d\cos(\Theta_s) = 0 + \int_0^t (\cos(\Theta_s)^2 + \sin(\Theta_s)^2)\,d\Theta_s = \int_0^t 1\,d\Theta_s = \Theta_t,$$

so $\Theta_t = B_t$!