Solutions 2

1. a) We have

$$\begin{aligned} X_t &= \sin(B_t) = \int_0^t \cos(B_s)\, dB_s - \frac{1}{2}\int_0^t \sin(B_s)\, ds, \\ Y_t &= e^{t/2}\sin(B_t) = \frac{1}{2}\int_0^t e^{s/2}\sin(B_s)\, ds + \int_0^t e^{s/2}\cos(B_s)\, dB_s - \frac{1}{2}\int_0^t e^{s/2}\sin(B_s)\, ds \\ &= \int_0^t e^{s/2}\cos(B_s)\, dB_s, \end{aligned}$$

so Y is martingale, but X is not.

b) $\langle X\rangle_t = \int_0^t \cos(B_s)^2\, ds$, $\langle Y\rangle_t = \int_0^t e^s \cos(B_s)^2\, ds$,

$$\langle X, B\rangle_t = \int_0^t \cos(B_s)\, ds, \quad \langle Y, B\rangle_t = \int_0^t e^{s/2}\cos(B_s)\, ds \quad \text{and} \quad \langle X, Y\rangle_t = \int_0^t e^{s/2}\cos(B_s)^2\, ds.$$

2. a) By Ito-Doeblin's formula, we have

$$X_t^2 = 2\int_0^t X_s\, dX_s + \frac{1}{2}\int_0^t 2\, d\langle X\rangle_s = 2\int_0^t X_s\, dX_s + \langle X\rangle_t.$$

Now, $\langle X\rangle_t = \int_0^t \sigma(X_s)^2\, ds$, so

$$X_t^2 = 2\int_0^t X_s\, \mu(X_s)\, ds + 2\int_0^t X_s\, \sigma(X_s)\, dB_s + \int_0^t \sigma(X_s)^2\, ds.$$

In the case where $2x\,\mu(x) + \sigma(x)^2 = 0$ for all $x \in \mathbb{R}$, we therefore obtain

$$X_t^2 = 2\int_0^t X_s\, \sigma(X_s)\, dB_s,$$

which is a martingale.

b) In general,

$$\langle X\rangle_t = \int_0^t \sigma(X_s)^2\, ds \quad \text{and} \quad \langle X^2\rangle_t = 4\int_0^t X_s^2\, \sigma(X_s)^2\, ds.$$

3. a) Since

$$\int_0^t X_s \circ dY_s = \int_0^t X_s\, dY_s + \frac{1}{2}\langle X, Y\rangle_t \quad \text{and} \quad \int_0^t Y_s \circ dX_s = \int_0^t Y_s\, dX_s + \frac{1}{2}\langle Y, X\rangle_t$$

and since $\langle Y, X\rangle_t = \langle X, Y\rangle_t$, the integration by parts formula reads

$$X_t\, Y_t - X_0\, Y_0 = \int_0^t X_s \circ dY_s + \int_0^t Y_s \circ dX_s.$$

b) We have

$$M_t = \int_0^t X_s \, dY_s + \frac{1}{2} \langle X, Y \rangle_t = \int_0^t e^{B_s} \, dB_s - \frac{1}{2} \int_0^t e^{B_s} \, ds + \frac{1}{2} \langle e^B, B \rangle_t.$$

Since

$$e^{B_t} - 1 = \int_0^t e^{B_s} \, dB_s + \frac{1}{2} \int_0^t e^{B_s} \, ds,$$

we obtain that $\langle e^B, B \rangle_t = \int_0^t e^{B_s} \, ds$, so

$$M_t = \int_0^t e^{B_s} \, dB_s - \frac{1}{2} \int_0^t e^{B_s} \, ds + \frac{1}{2} \int_0^t e^{B_s} \, ds = \int_0^t e^{B_s} \, dB_s$$

is indeed a martingale (one can check that the technical condition $\mathbb{E}(\int_0^t \exp(2B_2) \, ds) < \infty$ is verified for all $t \in \mathbb{R}_+$).