Solutions 11

1. a) $\langle M\rangle_t = \int_0^t s^2\,ds = t^3/3$.

b) $\tau(s) = \inf\{t > 0 \,:\, t^3/3 \geq s\} = \inf\{t > 0 \,:\, t \geq (3s)^{1/3}\} = (3s)^{1/3}$.

c) $\mathbb{E}(W_{s_2} - W_{s_1}) = \mathbb{E}(M((3s_2)^{1/3}) - M((3s_1)^{1/3})) = 0$, since M is a stochastic integral. Using now the isometry property, we have:

$$\begin{aligned}\mathbb{E}((W_{s_2} - W_{s_1})^2) &= \mathbb{E}((M((3s_2)^{1/3}) - M((3s_1)^{1/3}))^2) = \mathbb{E}\left(\left(\int_{(3s_1)^{1/3}}^{(3s_2)^{1/3}} s\,dB_s\right)^2\right) \\ &= \int_{(3s_1)^{1/3}}^{(3s_2)^{1/3}} s^2\,ds = s_2 - s_1.\end{aligned}$$

d) Being a Wiener integral, M is a Gaussian process. As the time change $\tau(s)$ is deterministic, W is also Gaussian. One can check further that W is actually a standard Brownian motion.

2. a) No. We have already seen that the process

$$M_t = \int_0^t \operatorname{sgn}(B_s)\,dB_s$$

on the right hand-side is a standard Brownian motion, since it is a continuous local martingale with quadratic variation $\langle M\rangle_t = t$. The process on the left-hand side being non-negative, the equality cannot hold.

b) Remembering that B has either $+\infty$ or $-\infty$ slope, the fact that L_t is non-zero is surprising: a process that moves with infinite speed either up or down should not spend too much time in a given place (namely $x = 0$ here). Nevertheless, the Brownian motion is full of surprises...

c) Let us first compute

$$\begin{aligned}\mathbb{E}(|B_t|) &= \int_{\mathbb{R}} dx\,|x|\,\frac{1}{\sqrt{2\pi t}}\,\exp\left(-\frac{x^2}{2t}\right) = \sqrt{\frac{2}{\pi t}}\int_0^\infty dr\,r\,\exp\left(-\frac{r^2}{2t}\right) \\ &= \sqrt{\frac{2}{\pi t}}\left(-t\,\exp\left(-\frac{r^2}{2t}\right)\right)\Bigg|_{r=0}^{r=\infty} = \sqrt{\frac{2t}{\pi}}.\end{aligned}$$

On the right-hand side, the first term has expectation zero. Let us then compute, permuting limit, integral and expectation:

$$\mathbb{E}(L_t) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t ds\,\mathbb{E}\left(1_{\{|B_s|<\varepsilon\}}\right)\,ds = \int_0^t ds\,\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\,\mathbb{P}(\{|B_s| < \varepsilon\})\,ds.$$

Notice that

$$\mathbb{P}(\{|B_s| < \varepsilon\}) = \int_{-\varepsilon}^{\varepsilon}\frac{1}{\sqrt{2\pi s}}\,\exp\left(-\frac{x^2}{2s}\right)\,dx,$$

so by the mean value theorem,

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mathbb{P}(\{|B_s| < \varepsilon\}) = \frac{1}{\sqrt{2\pi s}}, \quad \forall s > 0.$$

Finally, we obtain that

$$\mathbb{E}(L_t) = \int_0^t ds \, \frac{1}{\sqrt{2\pi s}} = \sqrt{\frac{2t}{\pi}},$$

which coincides with $\mathbb{E}(|B_t|)$ computed above.

NB: You may have noticed that $\mathbb{E}(L_t) > t$ for t sufficiently small. This indicates that L_t is not exactly the time spent by B in $x = 0$ on the interval $[0, t]$. Actually, L_t is only the *density* evaluated in $x = 0$ of the occupation measure of the process B. As such, it may exceed t, exactly like the density $p(x)$ of a continuous random variable may exceed 1 in $x = 0$ or elsewhere.