Solutions 10

1. a) We have

$$f'_{x_i}(\underline{x}) = -\frac{x_i}{\|\underline{x}\|^3} \quad \text{and} \quad f''_{x_i,x_i}(\underline{x}) = -\frac{1}{\|\underline{x}\|^3} + \frac{3\,x_i^2}{\|\underline{x}\|^5},$$

so $\Delta f(\underline{x}) = \sum_{i=1}^3 f''_{x_i,x_i}(\underline{x}) = 0$, for all $\underline{x} \neq 0$.

b) The result in a) implies that X is a local martingale.

c*) Let us compute

$$\mathbb{E}\left(\frac{1}{\|B_t\|}\right) = \int_{\mathbb{R}^3} d^3\underline{x}\,\frac{1}{\|\underline{x}\|\,(2\pi t)^{3/2}}\,\exp\left(-\frac{\|x\|^2}{2}\right) = C_t \int_0^\infty dr\, r^2\,\frac{1}{r}\,\exp(-r^2/2) < \infty,$$

where C_t is some constant (we use here the spherical coordinates to perform the integration).

d) By Ito-Doeblin's formula and the computation in a),

$$X_t = X_0 - \sum_{i=1}^3 \int_0^t \frac{B_s^{(i)}}{\|\underline{B}_s\|^3}\, dB_s^{(i)} + 0,$$

so

$$\langle X \rangle_t = \sum_{i=1}^3 \int_0^t \frac{(B_s^{(i)})^2}{\|\underline{B}_s\|^6}\, ds = \int_0^t \frac{1}{\|\underline{B}_s\|^4}\, ds.$$

Therefore, $\mathbb{E}(\langle X \rangle_t)$ is proportional to (using again spherical coordinates)

$$\int_0^t ds \int_0^\infty dr\, r^2\,\frac{1}{r^4}\,\exp(-r^2/2),$$

which is infinite (the integral in r diverges close to $r = 0$), so the answer to the question in the problem set is no.

e) At time τ_a, $\|\underline{B}_{\tau_a}\| = a$, so $X_{\tau_a} = 1/a$. Therefore, $\mathbb{E}(X_{\tau_a}) = 1/a$.

f) Part d) says that X is not a square-integrable martingale. Part e) says that the optional theorem is not applicable here, as $E(X_{\tau_a}) > \mathbb{E}(X_{\tau_b})$ for $a < b$, i.e., $\tau_a < \tau_b$. So this leads us to think that X is not a martingale (which is actually the case).

2. a) $M_t + nt = f(\underline{B}_t)$, where $f(\underline{x}) = \|\underline{x}\|^2$. Using Ito-Doeblin's formula, we therefore have

$$M_t + nt = M_0 + \sum_{i=1}^n \int_0^t 2B_s^{(i)}\, dB_s^{(i)} + \frac{1}{2}\sum_{i=1}^n \int_0^t 2\, ds,$$

i.e.,

$$M_t = \sum_{i=1}^n \int_0^t 2B_s^{(i)}\, dB_s^{(i)}$$

is indeed a local martingale.

b) For all $t \in \mathbb{R}_+$, we have

$$\mathbb{E}(\langle M \rangle_t) = \sum_{i=1}^n \int_0^t \mathbb{E}((B_s^{(i)})^2)\, ds = nt < \infty,$$

so M is a square-integrable martingale.

c) There is a hole in the rigorous argument here, whose repair would require more notions than the ones seen in class, so let me skip the proof.

d) Using the optional stopping theorem, we obtain

$$0 = \mathbb{E}(M_0) = \mathbb{E}(M_{\tau_a}) = \mathbb{E}(\|\underline{B}_{\tau_a}\|^2 - n\,\tau_a) = a^2 - n\,\mathbb{E}(\tau_a),$$

i.e., $\mathbb{E}(\tau_a) = a^2/n$. This result is of the same flavor as the one obtained in Homework 9, Exercise 2.

3. a) Using Ito-Doeblin's formula, we obtain:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X \rangle_s \\ &= \int_0^t \left(\frac{f'(X_s)\,(n-1)}{2X_s} + \frac{f''(X_s)}{2} \right) ds + \int_0^t f'(X_s)\, dW_s, \end{aligned}$$

as $\langle X \rangle_t = t$. So in order for $f(X_t)$ to be a local martingale, we need that

$$f''(x) = -\frac{(n-1)\,f'(x)}{x}, \quad \text{with the conditions } f(1) = 0,\ f'(1) = 1.$$

Writing $u(x) = f'(x)$, we obtain:

$$\frac{u'(x)}{u(x)} = -\frac{n-1}{x}, \quad u(1) = 1,$$

i.e., $\ln(u(x)) = (1-n)\,\ln(x)$, i.e., $u(x) = x^{-(n-1)}$. Therefore, if $n = 2$, we obtain $f(x) = \log(x)$, whereas if $n \geq 3$, we obtain $f(x) = (1 - x^{-(n-2)})/(n-2)$.

b) For all $n \geq 2$, the quadratic variation of Y is given here by

$$\langle Y \rangle_t = \int_0^t f'(X_s)^2\, ds = \int_0^t \frac{1}{\|\underline{B}_s\|^{2(n-1)}}\, ds.$$

$\mathbb{E}(\langle Y \rangle_t)$ is therefore proportional to (use again spherical coordinates)

$$\int_0^t ds \int_0^\infty dr\, r^{n-1}\, \frac{1}{r^{2(n-1)}}\, \exp(-r^2/2)$$

which is infinite, as the integral near $r = 0$ diverges (even for $n = 2$). So Y is certainly not a square-integrable martingale (but one can check with a little extra effort that it is neither a martingale).