Solutions 1

0*. a) If $t \geq s$, then $\mathrm{Cov}(M_t, M_s) = \mathbb{E}(M_t\, M_s) - \mathbb{E}(M_t)\,\mathbb{E}(M_s) = \mathbb{E}(\mathbb{E}(M_t \,|\, \mathcal{F}_s)\, M_s) - \mathbb{E}(M_0)^2 = \mathbb{E}(M_s^2) - \mathbb{E}(M_0^2)$ is only a function of s. So in the general case, $\mathrm{Cov}(M_t, M_s)$ is only a function of $t \wedge s$.

b) $\mathrm{Var}(M_t) = \mathbb{E}(M_t^2) - \mathbb{E}(M_t)^2 \geq \mathbb{E}(M_s^2) - \mathbb{E}(M_s)^2 = \mathrm{Var}(M_s)$, since (M_t^2) is a submartingale.

c) By assumption, $N_t - M_t \geq 0$ a.s., for all $t \in \mathbb{R}_+$. But since both M and N are martingales, $\mathbb{E}(N_t - M_t) = \mathbb{E}(N_0 - M_0) = 0$, also by assumption. So $N_t - M_t = 0$ a.s., for all $t \in \mathbb{R}_+$.

1. a) $\langle M \rangle_t = \int_0^t s^2\, ds = t^3/3$.

b) $\langle N \rangle_t = \int_0^t s^2\, d\langle M \rangle_s$. Since $\langle M \rangle_s = s^3/3$, $d\langle M \rangle_s = s^2\, ds$ and thus, $\langle N \rangle_t = \int_0^t s^4\, ds = t^5/5$.

c) $f(s) = s^2$. Indeed, $\langle \int_0^\cdot s^2\, dB_s \rangle_t = \int_0^t s^4\, ds = t^5/5$ is equal to the quadratic variation of N.

NB: $M_t = \int_0^t s\, dB_s$ may be rewritten in differential form as $dM_s = s\, dB_s$, so $N_t = \int_0^t s\, dM_s = \int_0^t s^2\, dB_s$.

2. a) $\langle M \rangle_t = \int_0^t e^{2s}\, ds = (e^{2t} - 1)/2$.

b) $\langle N \rangle_t = \int_0^t e^{-2s}\, d\langle M \rangle_s$. Since $\langle M \rangle_s = (e^{2s} - 1)/2$, $d\langle M \rangle_s = e^{2s}\, ds$ and thus, $\langle N \rangle_t = \int_0^t e^{2s}\, e^{-2s}\, ds = \int_0^t 1\, ds = t$.

c) Since $\langle N \rangle_t = t$, N is the Brownian motion B.

3. a) $\langle B, M \rangle_t = \int_0^t 1\, e^s\, ds = e^t - 1$.

b) $\mathbb{E}(B_t M_t) = \mathbb{E}(\langle B, M \rangle_t) = e^t - 1$.

c) $\langle M, N \rangle_t = \int_0^t e^s\, e^{-s}\, ds = t$.

d)
$$\mathbb{E}\left(\int_0^t s\, dM_s \int_0^t s\, dN_s\right) = \mathbb{E}\left(\int_0^t s^2\, d\langle M, N \rangle_s\right) = \int_0^t s^2\, ds = \frac{t^3}{3}.$$