

Final Exam

SURNAME: FIRST NAME: SECTION:

Authorized material: 4 handwritten one-sided A4 pages.

Exam duration: 4 hours.

Exercise 1. Quiz.Let $\sigma > 0$, let B be a standard Brownian motion and let X be the process defined as

$$X_t = \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right), \quad t \in \mathbb{R}_+.$$

a) Is X a martingale? yes nob) What is the SDE satisfied by X ?Answer: $dX_t = \dots\dots\dots$ $X_0 = \dots\dots\dots$ c) Is X a Gaussian process? yes nod) Compute the mean and the covariance of X .Answer: $\mathbb{E}(X_t) = \dots\dots\dots$ $\text{Cov}(X_t, X_s) = \dots\dots\dots$ e) Is the covariance of X a function of $t \wedge s$ only? yes nof) Compute the quadratic variation of X . Answer: $\langle X \rangle_t = \dots\dots\dots$ g) Is the quadratic variation of X deterministic? yes noh) Does X have independent increments? yes noi) Does it hold that (*) $\mathbb{E}(\langle X \rangle_t) = \mathbb{E}(X_t^2) - \mathbb{E}(X_0^2)$, $\forall t \in \mathbb{R}_+$? yes noj) Does eq. (*) hold for *any* continuous square-integrable martingale X ? yes nok) Does eq. (*) hold for *any* continuous semi-martingale X (whose martingale part is a continuous square-integrable martingale)? yes no

Exercise 2. Let $\mu \in \mathbb{R}$, $\sigma > 0$ and let B be a standard Brownian motion. Let also X be the strong solution of the following SDE:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad X_0 = 1,$$

and let Y be the process defined as $Y_t = \frac{1}{X_t}$, $t \in \mathbb{R}_+$.

- a) Compute the SDE satisfied by the process Y .
- b) Does there exist values of μ and σ for which both the processes X and Y are simultaneously submartingales? If yes, then describe the range of parameter values for which this happens. If no, then explain why this cannot happen.
- c) Does there exist values of μ and σ for which both the processes X and Y are simultaneously martingales? If yes, then describe the range of parameter values for which this happens. If no, then explain why this cannot happen.
- d) Compute the quadratic covariation of X and Y .
- e) Is the process $(X_t Y_t - \langle X, Y \rangle_t, t \in \mathbb{R}_+)$ is a martingale? Justify your answer.

Exercise 3. Let B be a standard (one-dimensional) Brownian motion and let us consider the following two-dimensional SDE:

$$\begin{cases} dX_t = \frac{X_t}{2} dt + Y_t dB_t, & X_0 = 0, \\ dY_t = \frac{Y_t}{2} dt + X_t dB_t, & Y_0 = 1. \end{cases}$$

- a) A priori, does there exist a unique strong solution (X, Y) to this SDE? Justify your answer.
- b) Write the two-dimensional SDE satisfied by the process (U, V) defined as

$$U_t = X_t + Y_t, \quad V_t = Y_t - X_t, \quad t \in \mathbb{R}_+.$$
- c) Solve this new SDE and deduce an analytic expression for the process (X, Y) .
- d) Show that $Y_t^2 = 1 + X_t^2, \forall t \in \mathbb{R}_+$.
- e) Does there exist a probability measure under which the process (X, Y) is a two-dimensional martingale? Justify your answer.

Exercise 4. Let $x_0 > 0$, let B be a standard Brownian motion and let X be the strong solution of the following SDE:

$$dX_t = +\frac{1}{2} X_t dt + X_t dB_t, \quad X_0 = x_0.$$

a) Let Y be the process defined as $Y_t = f(t, X_t)$, where $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. What condition on f ensures that the process Y is a local martingale?

Let us now assume that f satisfies this condition and let us take for granted that this condition also implies that the process Y is a martingale. Let us moreover assume that $f(T, x) = h(x)$ for all $x \in \mathbb{R}$, for some $T > 0$ and $h \in C(\mathbb{R})$.

b) Deduce from the above a stochastic representation for the function $f(0, x_0)$.

From now on, let us consider the particular case where $h(x) = x^2$.

c) In this particular case, compute what the function $f(0, x_0)$ is.

d) Assume now that X represents the evolution in time of a stock price and that $h(X_T)$ represents the payoff of a European option on the stock X at maturity T . Is $f(0, x_0)$ the right price at time $t = 0$ for the premium of such an option? If yes, then justify your answer. If no, then compute what the right price is.

e) What is the right hedging strategy at time $t = 0$ for such an option? I.e., what is the right amount of stock ϕ_0 to buy at time $t = 0$ in order to start hedging the option? (We do not consider here the hedging strategy over the whole period $[0, T]$.)

Exercise 5. Let $c > 0$, let B be a standard Brownian motion and let $\tau = \inf\{t > 0 : B_t \geq c\}$.

a) Does there exist a constant $K > 0$ such that $\tau \leq K$ a.s.?

b) Is $\mathbb{P}(\tau < \infty) = 1$?

c) Is it true that $\mathbb{E}(B_\tau) = \mathbb{E}(B_0)$? Justify your answer.

Let now X be the process defined as $\left(X_t = \frac{1}{c-B_t}, 0 < t < \tau\right)$ and let us define the increasing sequence of stopping times $\tau_n = \inf\{t > 0 : B_t \geq c - \frac{1}{n}\}$, $n \geq 1$.

d) For each $n \geq 1$, what type of process is $X^{\tau_n} = (X_{t \wedge \tau_n}, t \in \mathbb{R}_+)$?

e) Compute the SDE satisfied by the process X (up to time τ).

f) Compute the quadratic variation of X (up to time τ).