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Abstract

The framework of this dissertation is the study of wave propagation phenomena, where the waves considered are generated by noise sources which are random both in time and space. More precisely, we are interested in finding real-valued solutions of linear hyperbolic partial differential equations (typically the wave equation) driven by additive Gaussian noise sources which are white in time and concentrated on surfaces in space.

It is a well known fact that when the spatial dimension is greater than one, the wave equation driven by a space-time Gaussian white noise admits a solution which takes its values in a distribution space. If we want the solution to be function-valued, it is natural to consider noise with some spatial correlation.

In this dissertation, we are going to see that this also happens for the wave equation driven by a Gaussian noise concentrated on a surface, in particular the sphere or the plane. In both cases, we give minimal conditions on the spatial covariance of the noise which guarantee the existence of a function-valued solution of the linear equation.

For the case of a noise concentrated on a d-dimensional sphere, we give two conditions, one necessary and one sufficient, for the existence of a square-integrable solution of the linear wave equation in the ball delimited by the sphere. These conditions are expressed in terms of the Fourier coefficients of the spatial covariance of the noise. In the case of a noise concentrated on a circle, the necessary condition can be reformulated into an explicit condition on the covariance.

For the linear wave equation in a d-dimensional space driven by noise concentrated on a k-plane (with $1 \leq k < d$), we give optimal conditions for the existence of a solution with values in some fractional Sobolev space in the directions perpendicular to the k-plane. These conditions are expressed in terms of the spectral measure of the noise and can also be reformulated into explicit conditions on the covariance.

Moreover, for the particular case of a noise concentrated on a hyperplane, we give two optimal conditions on the spectral measure which guarantee the existence of a real-valued solution defined, respectively, only outside the hyperplane that supports the noise, or everywhere in space. In the latter case, we establish the existence and uniqueness of a real-valued solution for a non-linear equation of the same type. Under stronger conditions on the spectral measure, we also establish that the solution of the linear equation is Hölder-continuous outside the hyperplane.

Finally, we consider similar questions for the linear heat equation in a d-dimensional space driven by noise concentrated on a k-plane (with $1 \leq k < d$) and we show that under a fairly mild assumption on the covariance of the noise, there exists a real-valued solution which is defined outside the hyperplane, but the solution is never defined on the k-plane itself.

Version abrégée

Le cadre de cette thèse est l'étude des phénomènes de propagation d'ondes engendrées par des sources de bruit aléatoires en temps et en espace. Plus précisément, nous nous intéressons à l'existence de solutions à valeurs réelles d'équations linéaires aux dérivées partielles hyperboliques (du type de l'équation des ondes) dirigées par des sources de bruit gaussiennes additives, décorrélées en temps et concentrées sur des surfaces en espace.

Il est bien connu que lorsque la dimension spatiale considérée est supérieure à un, l'équation d'onde linéaire dirigée par un bruit gaussien blanc en temps et en espace admet une solution qui prend ses valeurs dans un espace de distributions. Si l'on veut obtenir une solution à valeurs dans un espace de fonctions, il est naturel de considérer que le bruit possède une certaine corrélation spatiale.

Dans cette thèse, nous allons voir que ceci se produit également pour l'équation d'onde linéaire dirigée par un bruit gaussien concentré sur une surface, en particulier la sphère ou le plan. Pour ces deux cas, nous donnons des conditions minimales sur la covariance spatiale du bruit garantissant l'existence d'une solution à valeurs dans un espace de fonctions pour l'équation linéaire.

Pour le cas d'un bruit concentré sur une sphère d-dimensionnelle, nous donnons deux conditions, une nécessaire et une suffisante, pour l'existence d'une solution de carré intégrable de l'équation d'onde linéaire dans la boule délimitée par la sphère. Ces conditions sont exprimées en termes des coefficients de Fourier de la covariance spatiale du bruit. Dans le cas d'un bruit concentré sur un cercle, la condition nécessaire peut être reformulée en une condition explicite sur la covariance.

Pour l'équation d'onde linéaire dans l'espace de dimension d dirigée par un bruit concentré sur un k-plan (avec $1 \leq k < d$), nous donnons des conditions optimales pour l'existence d'une solution à valeurs dans un espace de Sobolev fractionnaire dans les directions perpendiculaires au k-plan. Ces conditions sont exprimées en termes de la mesure spectrale du bruit et peuvent aussi être reformulées en des conditions explicites sur la covariance.

De plus, pour le cas particulier d'un bruit concentré sur un hyperplan, nous donnons deux conditions sur la mesure spectrale garantissant l'existence d'une solution à valeurs réelles définie respectivement seulement en dehors de l'hyperplan qui supporte le bruit, ou partout dans l'espace. Dans ce dernier cas, nous montrons l'existence et l'unicité d'une solution à valeurs réelles pour une équation non-linéaire du même type. Sous des conditions plus fortes concernant la mesure spectrale, nous montrons également que la solution de l'équation linéaire est höldérienne en dehors de l'hyperplan.

Finalement, nous considérons des questions similaires pour l'équation de la chaleur linéaire dans l'espace de dimension d dirigée par un bruit concentré sur un k-plan (avec $1 \leq k < d$) et nous montrons que sous une hypothèse peu restrcitive concernant la covariance du bruit, il existe une solution à valeurs réelles définie en dehors du k-plan, mais cette solution n'est jamais définie sur le k-plan lui-même.

Contents

Chapter 1

Introduction

Stochastic partial differential equations are of both mathematical and practical interest. The mathematical aspects concern the extension of the now well developed theory of partial differential equations to similar equations with random source terms, which are strongly irregular, both in time and space. On the other hand, the practical interest of these equations comes from the fact that they provide models for physical phenomena with temporal and spatial variations that are too rapid to be well described by deterministic models. Examples of such phenomena are to be found in various domains, such as oceanography [3, 22], fluid mechanics [10] or mathematical finance [8].

Many approaches have been developed in order to handle these new kinds of equations. In the present dissertation, we follow mainly the approach described by J. B. Walsh in [62], which considers partial differential equations driven by additive noises (essentially white noise). Solutions of such equations are described as *random fields* indexed by the time and space variables, and are expressed as generalized stochastic integrals with respect to a *martingale measure* constructed from the noise under consideration. For the same kind of equations, one could also use the approach of G. Da Prato and J. Zabczyk described in [18, 19], which consider solutions as processes indexed by the time variable with values in some functional space of the space variable, namely Banach or Hilbert spaces. We will also use some aspects of this approach in the present dissertation. For different approaches to stochastic partial differential equations, see [10, 27, 31, 48], among many others. Let us also mention here that there is another "class" of stochastic partial differential equations with no additive noise but random coefficients, which has lead to different types of analysis and results (see for example [47, 60]).

The first kind of stochastic partial differential equations which have been studied are those driven by additive space-time Gaussian white noise. For the linear equation, viewing the noise as a random distribution allows to apply deterministic methods; see [28, 37, 63]. In the specific case where the spatial dimension is one, it can be shown that even though the noise is strongly irregular, the solution of the equation is continuous; it is therefore possible to consider non-linear equations of the same type: see for example [11, 12, 49, 50]. On the other hand, the solution of the linear equation is no longer a function when the space dimension is greater than one. One possible way of studying non-linear equations in higher space dimension is then to define the non-linear transformation of a distribution (see [43, 44]). Another way is to replace the white noise by a spatially homogeneous Gaussian noise whose covariance satisfies a minimal condition in order for the solution to be a function, and then to analyze non-linear equations of the same type. This latter approach is reviewed in the next section.

The main purpose of the present dissertation is to study hyperbolic partial differential equations driven by Gaussian noises that are white in time and concentrated in space on manifolds of lower dimension than the space variable under consideration; typically a sphere or a plane. It turns out that also in this case, solutions are not function-valued when we consider *white* noises concentrated on surfaces and when the space dimension is greater than one. Since we do not want to consider non-linear transformations of distributions, we will follow the second approach described above, and therefore try to find minimal conditions on the covariance of the noise in order to obtain function-valued solutions. This allows us afterwards to study equations driven by non-linear stochastic source terms concentrated on these manifolds.

A typical example of such an equation is the equation describing wave propagation in ordinary three dimensional space perturbed by a noise concentrated on a plane. This type of situation might for example arise in the study of the sound wave produced by the noise of the rain falling on the surface of a lake. This noise is composed of a large number of small contributions (namely the droplets of rain); it is therefore natural to consider that it is Gaussian. Moreover, it is concentrated on a surface (namely the lake surface), so the pressure wave emitted by this noise satisfies a wave equation driven by an additive noise source concentrated on a plane.

There have been many studies of equations driven by noises concentrated on manifolds (generally considered as stochastic boundary conditions). Most of these studies however concern the case where the spatial dimension is one, so the boundary noise is therefore a pointwise noise (see for example [4, 20, 33]). Some recent results have been obtained for *parabolic* equations driven by boundary noises in higher space dimension: see [34, 61]. Among these two papers, the most closely related to this dissertation is the paper [61]. The methods for parabolic equations differ from those studied here, since these equations exhibit regularizing properties, which is not the case of hyperbolic equations. This has made it possible to analyse parabolic equations driven by noises concentrated on fairly general manifolds. For the case of hyperbolic equations, the analysis is more intricate, because we need Fourier analysis techniques developed for equations driven by spatially homogeneous noises. In the present dissertation, we therefore restrict ourselves to equations driven by noises concentrated on two canonical manifolds: the sphere and the plane.

1.1 Equations driven by spatially homogenous noises

Even though a noise concentrated on a manifold is not spatially homogeneous, the techniques that we use in this dissertation are quite similar to those used for equations driven by spatially homogeneous noises. The first contributions to this subject are to be found in [13, 17, 21, 26, 38, 39, 40]. Afterwards, there have been many papers on the subject, some following the approach of J.B. Walsh (see [14, 15, 35, 36, 54, 55]) and some the approach of G. Da Prato and J. Zabczyk (see [30, 51, 52, 53]).

In this section, we describe the recent results obtained independently by R. C. Dalang in [15] and A. Karczewska and J. Zabczyk in [30] for the heat and the wave equation in $\mathbb{R}^d$ $(d \geq 1)$ driven by spatially homogeneous noise. The equation considered in these references is the following:

$$Lu(t,x) = \dot{F}(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where L is either the heat or the wave operator (with vanishing initial conditions) and $\dot{F}$ is a generalized centered Gaussian process with covariance

$$\mathbb{E}(\dot{F}(t,x)\ \dot{F}(s,y)) = \delta_0(t-s)\ \Gamma(x-y),$$

where δ_0 is the usual Dirac measure on $\mathbb{R}$ and Γ is a non-negative and non-negative definite tempered Borel measure on $\mathbb{R}^d$. Let us also denote by μ the spectral measure of the noise, defined as the inverse Fourier transform of Γ.

Theorems 11 and 13 in [15], as Theorem 1 in [30], state that there exists a real-valued process which is the solution (in a weak sense) of the above equation if and only if the following condition on μ is satisfied:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty.$$

The reformulation of this condition into a condition on Γ yields (see Theorem 2 in [30]):

$$\begin{cases} \text{no condition on } \Gamma, & \text{when d=1,} \\ \displaystyle\int_{B(0,1)} \Gamma(dx)\ \ln\left(\frac{1}{|x|}\right) < \infty, & \text{when } d=2, \\ \displaystyle\int_{B(0,1)} \Gamma(dx)\ \frac{1}{|x|^{d-2}} < \infty, & \text{when } d \geq 3. \end{cases}$$

When $\Gamma(dx) = f(|x|)\ dx$, with f a continuous function on $]0,\infty[$, this implies the following.

- When $d=2$, the above condition is equivalent to

$$\int_0^1 dr\ f(r)\ r\ \ln\left(\frac{1}{r}\right) < \infty.$$

- When $d \geq 3$, the above condition is equivalent to

$$\int_0^1 dr\ f(r)\ r < \infty.$$

Note that the condition for $d = 2$ was previously obtained by R.C. Dalang and N. Frangos in [14].

Moreover, Theorem 13 in [15] states that when $d \leq 3$, under the same assumption on μ and for globally Lipschitz functions g and h, there exists a real-valued process which is the solution of the following non-linear equation:

$$Lu(t,x) = g(u(t,x)) + h(u(t,x))\ \dot{F}(t,x).$$

The study of the regularity of this real-valued solution has been performed by M. Sanz-Solé and M. Sarrà in [54] for the semi-linear equation, that is, when $h \equiv 1$, and the result is the following. If $\beta \in \,]0,1[$ and

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{1-\beta}} < \infty,$$

Theorems 4.1 and 4.2 in [54] then state that the solution of the equation is a.s. locally Hölder-continuous with exponent $\gamma < \beta$ in x, and $\gamma < \beta/2$ in t in the case of the heat equation or $\gamma < \beta \wedge \frac{1}{2}$ in t in the case of the wave equation. For other results on the regularity of the solution of this kind of equations, see also [35, 36, 55].

Finally, let us also mention that another approach to these non-linear equations can be found in [51, 52, 53], where it is shown that the solution belongs to some weighted Sobolev space on $\mathbb{R}^d$.

1.2 Equations driven by boundary noises

In this section, we summarize briefly the main results obtained in the present dissertation, focusing on the results concerning hyperbolic equations driven by boundary noises, which is our main interest. We also describe briefly what are the main ideas of the proofs of these results, reviewing the chapters of the dissertation at the same time.

Noise on a sphere (Chapters 2 and 3). Let $d \geq 2$ and consider the equation

$$L_{a,b}u(t,x) = \dot{F}(t,x)\ \delta_{\partial B(0,1)}(x), \qquad (t,x) \in \mathbb{R}_+ \times B(0,1),$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^d$ and $\partial B(0,1)$ is the unit sphere in $\mathbb{R}^d$, $a, b \in \mathbb{R}$ and $L_{a,b}$ is the following linear partial differential operator

$$L_{a,b} = \frac{\partial^2}{\partial t^2} + 2a\,\frac{\partial}{\partial t} + b - \Delta,$$

with homogeneous Neumann boundary conditions on $\partial B(0,1)$ (note that when $a = b = 0$, $L_{a,b}$ is simply the wave operator). The noise $\dot F$ considered here is a Gaussian centered noise with covariance

$$\mathbb{E}(\dot F(t,x)\ \dot F(s,y)) = \delta_0(t-s) \sum_{l\in\mathbb{N}} a_l\ P_l(x\cdot y), \qquad x,y \in \partial B(0,1), \tag{1.1}$$

where the a_l are non-negative numbers and the P_l are the generalized Legendre polynomials. That is, $\dot F$ is a boundary noise, white in time and rotationally invariant on the sphere.

The result is then the following (see Theorem 3.3.3): a sufficient condition for the existence of an $L^2(B(0,1))$-valued process $\{u(t),\ t \in \mathbb{R}_+\}$ which is a solution (in a weak sense) of the above equation is

$$\sum_{l\in\mathbb{N}} \frac{a_l\ \ln(1+l)}{1+l} < \infty,$$

and a necessary condition is

$$\sum_{l\in\mathbb{N}} \frac{a_l}{1+l} < \infty.$$

In order to establish this result, we consider in Chapter 2 the hyperbolic equation in a bounded domain driven by general noise. We first establish a weak formulation of the equation and prove by standard techniques using the spectral decomposition of the Laplacian in a bounded domain, that under a specific condition on the covariance of the noise, there exists a unique weak solution which is a process $u = \{u(t),\ t \in \mathbb{R}_+\}$ with values in $L^2(D)$.

In Chapter 3, we consider more specifically the case where the domain is the unit ball $B(0,1)$ in $\mathbb{R}^d$ and the noise is concentrated on the sphere $\partial B(0,1)$, with covariance of the form given in (1.1), which can be shown to be a fairly general form. We then particularize the specific condition obtained in Chapter 2 to the present case, which brings us to the above mentioned result. The proof relies mainly on estimates on the eigenvalues and eigenvectors of the Laplacian in the unit ball, which are expressed in terms of spherical harmonics and Bessel functions.

Noise on a k-plane (Chapters 4, 5 and 6). Let $d > k \geq 1$ and for $x \in \mathbb{R}^d$, write $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^{d-k}$. Consider the equation

$$L_{a,b}u(t,x) = \dot F(t,x_1)\ \delta_0(x_2), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1.2}$$

where $a, b \in \mathbb{R}$ and $L_{a,b}$ is the following linear partial differential operator on $\mathbb{R}^d$:

$$L_{a,b} = \frac{\partial^2}{\partial t^2} + 2a\ \frac{\partial}{\partial t} + b - \Delta.$$

The noise $\dot F$ considered here is a Gaussian centered noise with covariance

$$\mathbb{E}(\dot F(t,x_1)\ \dot F(s,y_1)) = \delta_0(t-s)\ \Gamma(x_1 - y_1),$$

where Γ is a non-negative definite measure on $\mathbb{R}^k$. That is, the noise $\dot{F}$ is white in time, concentrated on the k-plane $x_2 = 0$, and spatially homogeneous on this k-plane.

Let μ be the spectral measure of the noise (defined as the inverse Fourier transform of Γ in the coordinate x_1), $\beta < 1 - \frac{d-k}{2}$ and $H^\beta(\mathbb{R}^{d-k})$ be the fractional Sobolev space of order β on $\mathbb{R}^{d-k}$. We then have the following result (see Theorem 6.4.3): the solution of the above equation is an $H^\beta(\mathbb{R}^{d-k})$-valued process $\{u(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ if and only if

$$\begin{aligned} \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2} < \infty, \quad & \text{when } \beta < -\frac{d-k}{2}, \\ \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)\ \ln\left(1+|\xi_1|^2\right)}{1+|\xi_1|^2} < \infty, \quad & \text{when } \beta = -\frac{d-k}{2}, \\ \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{1+\frac{d-k}{2}-\beta}} < \infty, \quad & \text{when } \beta \in \left]-\frac{d-k}{2}, 1-\frac{d-k}{2}\right[. \end{aligned} \tag{1.3}$$

One can notice that β is possibly non-negative only when $k = d-1$, that is, when the noise is concentrated on a hyperplane.

In order to obtain this result, after some preliminaries in Chapter 4, we consider in Chapter 5 the hyperbolic equation in $\mathbb{R}^d$ driven by general noise and show that under a fairly mild assumption on the covariance of the noise, there always exists a unique weak solution which is a process $u = \{u(t),\ t \in \mathbb{R}_+\}$ with values in some distribution space. For this proof, we use the Walsh theory [62] of stochastic integrals with respect to martingale measures on $\mathbb{R}^d$.

In Chapter 6, we consider the case where the noise is concentrated on a k-plane, and first show that the Fourier transform of the weak solution in x_2 (that is, in the directions perpendicular to the k-plane) is a real-valued process (rather than a distribution-valued one) if and only if a specific condition on the spectral measure of the noise is satisfied, namely condition (1.3) (see Proposition 6.3.6). The scheme that we follow for this proof is similar to that used by R. Dalang for the equation driven by spatially homogeneous noise on $\mathbb{R}^d$ (see [15]). We first extend the Walsh stochastic integral to distribution-valued integrands and then show that under the above mentioned condition, it is possible to define a real-valued process which is the stochastic integral of a distribution-valued integrand (namely the Fourier transform in x_2 of the Green kernel of the equation) and which is the Fourier transform of the weak solution in x_2. We then show that if it is possible to define such a process, then the above condition is satisfied, proving therefore that this condition is optimal.

In order then to establish the above result, we use the classical fact that a distribution belongs to a fractional Sobolev space H^β if and only if its Fourier transform belongs to a weighted L^2-space (the weight depending on β). We then study the integrability of the square of the

Fourier transform of the solution in x_2, which we found to be a real-valued process in the preceding analysis. Some technical estimates are needed here, which lead to the above optimal conditions.

Noise on a hyperplane (Chapters 7 and 8). Consider the same equation as above in the case where $k = d - 1$. Our first two results are the following: Theorem 7.2.5 states that the solution of equation (1.2) is a real-valued process $\{u(t, x_1, x_2),\ (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*\}$ defined outside the hyperplane $x_2 = 0$ if and only if

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1 + |\xi_1|^2}} < \infty.$$

Theorem 7.3.3 then states that the solution of the above equation is a real-valued process $\{u(t, x),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ defined on the whole space (including the hyperplane $x_2 = 0$) if and only if the following stronger condition on μ is satisfied:

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)\ \ln\left(\sqrt{1 + |\xi_1|^2}\right)}{\sqrt{1 + |\xi_1|^2}} < \infty. \tag{1.4}$$

The fact that this condition is different from the previous one shows that it is quite different to require that a process solution be defined *everywhere* rather that just *almost* everywhere, such as for $x_2 \neq 0$.

In Chapter 7, we establish the first two mentioned results by techniques similar to those of Chapter 6. We also establish some regularity properties of the solution, using techniques similar to those used by M. Sanz-Solé and M. Sarrà in [54] for spatially homogeneous noises. Namely, we show (see Theorem 7.4.3) that the solution is a.s. locally Hölder-continuous with exponent $\gamma < \beta \in]0, \frac{1}{2}[$ outside the hyperplane $x_2 = 0$ if

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1 + |\xi_1|^2)^{\frac{1}{2} - \beta}} < \infty.$$

Furthermore, when $d \leq 3$ and $a^2 \geq b$, we also consider the following non-linear equation

$$Lu(t, x) = g(u(t, x_1, 0))\ \delta_0(x_2) + h(u(t, x_1, 0))\ \dot{F}(t, x_1)\ \delta_0(x_2).$$

Theorem 8.1.1 states that there exists a real-valued process u which is solution of this equation if g and h are globally Lipschitz functions and condition (1.4) is satisfied.

In Chapter 8, we follow the approach of R. Dalang developed in [15] for non-linear equations driven by spatially homogeneous noises. We first study a mild formulation of the non-linear equation described above, restricted to the hyperplane $x_2 = 0$, and show that this equation admits a unique solution by the standard Picard's iteration scheme used for spatially homogeneous noises. We then extend the solution to the whole space.

Let us finally mention the two following facts. At the end of Chapter 7, we establish that when $k = d - 2$, there does not exist a real-valued process defined outside the k-plane $x_2 = 0$ which is the solution of the linear hyperbolic equation driven by noise on the k-plane. On the other hand, we study in Chapter 9 the case of the heat equation in $\mathbb{R}^d$, which is the simplest example of a parabolic equation, driven by noise on a k-plane. The answers obtained for this equation are rather different than for the hyperbolic one. The two main results (see Theorems 9.2.5 and 9.2.7) state that under a fairly mild assumption on the covariance, the solution is always a real-valued process defined outside the k-plane $x_2 = 0$, but that on the other hand, it can never be defined on the k-plane itself. We then compare these results with those obtained by R. Sowers in [61].

Chapter 2

Linear equation in a bounded domain

Let D be a bounded domain in $\mathbb{R}^d$ whose boundary ∂D is a C^∞ manifold and such that D is locally on one side of ∂D. Let also $a, b \in \mathbb{R}$. We are interested in solving the following stochastic linear hyperbolic equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) + b\,u(t,x) - \Delta u(t,x) = \dot{F}^D(t,x), \qquad (t,x) \in \mathbb{R}_+ \times D, \\ \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \partial D, \\ \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in D, \end{cases} \tag{2.1}$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative of u at the boundary, u_0, v_0 are two given functions on D, and $\dot{F}^D = \{\dot{F}^D(t,x),\ (t,x) \in \mathbb{R}_+ \times D\}$ is a generalized centered Gaussian process whose covariance is formally given by

$$\mathbb{E}(\dot{F}^D(t,x)\ \dot{F}^D(s,y)) = \delta_0(t-s)\ \Gamma_D(x,y),$$

where δ_0 is the usual Dirac measure on $\mathbb{R}$ and Γ_D is a non-negative definite distribution on $D \times D$, in a sense that will be precised below.

2.1 Spectral theorem

Let us consider the following spaces:

- $\mathcal{S}(D) = \left\{ \varphi \in C^\infty(\overline{D}) \text{ such that } \dfrac{\partial \varphi}{\partial \nu}\Big|_{\partial D} = 0 \right\}$, the space of test functions.

- $L^2(D)$, the usual space of measurable and square integrable functions on D, equipped with the scalar product

$$\langle u, v \rangle_0 = \int_D dx\ u(x)\ v(x),$$

and the corresponding norm $\|\cdot\|_0$.

- $H^1(D)$, the Sobolev space of functions in $L^2(D)$ whose first partial derivatives belong also to $L^2(D)$, equipped with the scalar product

$$\langle u, v\rangle_1 = \langle u, v\rangle_0 + \langle \nabla u, \nabla v\rangle_0,$$

and the corresponding norm $\|\cdot\|_1$ (note that $\mathcal{S}(D) \subset H^1(D)$).

- $H^{-1}(D)$, the dual of $H^1(D)$, equipped with the norm

$$|||u|||_{-1} = \sup_{\varphi\in H^1(D),\varphi\neq 0} \frac{|u(\varphi)|}{\|\varphi\|_1}$$

Let us also denote by $\langle\cdot,\cdot\rangle_{-1,1}$ the *duality product* between $H^{-1}(D)$ and $H^1(D)$, simply defined by

$$\langle u, \varphi\rangle_{-1,1} = u(\varphi).$$

We will use the following theorem from classical analysis (see [59, pp. 111-112]), which states the existence of a Hilbertian basis of $L^2(D)$ composed by the eigenfunctions of the Laplacian operator on D with Neumann boundary conditions. Before stating it, let us write $a_n \sim b_n$ when there exists $C \in]0,\infty[$ such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C.$$

Theorem 2.1.1. *There exist two sequences* $\{e_n, n \in \mathbb{N}\} \subset \mathcal{S}(D)$ *and* $\{\lambda_n, n \in \mathbb{N}\} \subset \mathbb{R}_+$ *such that*

$$\Delta e_n + \lambda_n e_n = 0, \qquad \forall n \in \mathbb{N},$$

$\{e_n, n \in \mathbb{N}\}$ *is a Hilbertian basis of* $L^2(D)$ *and* $\{\lambda_n, n \in \mathbb{N}\}$ *is an increasing sequence of non-negative numbers such that*

$$\lambda_n \sim n^{2/d} \quad as \quad n \to \infty. \tag{2.2}$$

Note that $\lambda_0 = 0$ and $e_0(x) \equiv \frac{1}{\sqrt{|D|}}$, since we consider Neumann boundary conditions.

Estimate (2.2) is known as *Weyl's law.* In the following, we will only use a consequence of this estimate, namely that the eigenvalues of the Laplacian tend to infinity as $n \to \infty$.

A direct consequence of the above theorem is that

$$\left\{\frac{e_n}{\sqrt{1+\lambda_n}},\ n \in \mathbb{N}\right\}$$

is a Hilbertian basis of $H^1(D)$. Moreover, we have the following equalities:

$$\begin{aligned}\|u\|_0^2 &= \sum_{n\in\mathbb{N}} |\langle u, e_n\rangle_0|^2,\\ \|u\|_1^2 &= \sum_{n\in\mathbb{N}} (1+\lambda_n)\, |\langle u, e_n\rangle_0|^2,\end{aligned}$$

and the norm $|||\cdot|||_{-1}$ on $H^{-1}(D)$ is equivalent to

$$\|u\|_{-1}^2 = \sum_{n\in\mathbb{N}} \frac{|\langle u, e_n\rangle_{-1,1}|^2}{1+\lambda_n}.$$

2.2 Gaussian noise

Since $\dot{F}^D(t,x)$ is not well defined for fixed $(t,x) \in \mathbb{R}_+ \times D$, we will rather consider in the following the process $F^D = \{F_t^D(\varphi),\ t\in\mathbb{R}_+,\ \varphi\in\mathcal{S}(D)\}$ which is related to $\dot{F}^D$ by the informal relationship

$$F_t^D(\varphi) = \int_0^t ds \int_D dx\ \dot{F}^D(s,x)\ \varphi(x), \qquad t\in\mathbb{R}_+,\ \varphi\in\mathcal{S}(D). \tag{2.3}$$

In order to define F^D rigorously, we assume that the covariance Γ_D is a bilinear, symmetric and *non-negative definite form on* $\mathcal{S}(D)$, that is,

$$\sum_{i,j=1}^m c_i\ c_j\ \Gamma_D(\varphi_i,\varphi_j) \geq 0, \qquad \forall m\geq 1,\ c_1,\dots,c_m\in\mathbb{R},\ \varphi_1,\dots,\varphi_m\in\mathcal{S}(D).$$

By the Kolmogorov extension theorem, (see [42, prop. 3.4]), there exist a probability space $(\Omega,\mathcal{G},\mathbb{P})$ and a centered Gaussian process $F^D = \{F_t^D(\varphi),\ t\in\mathbb{R}_+,\ \varphi\in\mathcal{S}(D)\}$ defined on this space, whose covariance is given by

$$\mathbb{E}(F_t^D(\varphi)\ F_s^D(\psi)) = (t\wedge s)\ \Gamma_D(\varphi,\psi).$$

Moreover, there exists a modification $\tilde{F}^D = \{\tilde{F}_t^D(\varphi),\ t\in\mathbb{R}_+\}$ of F^D such that for all $\varphi\in\mathcal{S}(D)$, the process $\{\tilde{F}_t^D(\varphi),\ t\in\mathbb{R}_+\}$ is a $\mathbb{P}-a.s.$ continuous Brownian motion with covariance parameter $\Gamma_D(\varphi,\varphi)$. In the following, we will consider implicitely the modification $\tilde{F}^D$.

2.3 Weak formulation of the equation

Now that we have a precise definition of the Gaussian noise under consideration, we also need to give a rigorous meaning to equation (2.1). Setting formally $v(t,x) = \frac{\partial u}{\partial t}(t,x)$, we obtain the following two formal equations, after integration in t of equation (2.1):

$$\begin{cases} u(t,x) = u_0(x) + \displaystyle\int_0^t ds\ v(s,x), \\ v(t,x) = v_0(x) + \displaystyle\int_0^t ds\ (-2a\ v(s,x) - b\ u(s,x) + \Delta u(s,x) + \dot{F}^D(s,x)). \end{cases}$$

We now multiply both sides of these two equations by a test function $\varphi\in\mathcal{S}(D)$ and integrate them in x on the domain D, with two more integrations by parts in x of the term with the Laplacian, taking into account the fact that $\frac{\partial\varphi}{\partial\nu}|_{\partial D} = 0$ and the Neumann boundary condition $\frac{\partial u}{\partial\nu}|_{\partial D} = 0$. Assuming that $(u_0, uv0) \in L^2(D)\oplus H^{-1}(D)$, considering that (u,v) takes its values in $L^2(D)\oplus H^{-1}(D)$ and using the informal relationship (2.3) gives then the following rigorous formulation: a *weak solution* of equation (2.1) is a process $(u,v) = \{(u(t),v(t)),\ t \in \mathbb{R}_+\}$ with values in $L^2(D) \oplus H^{-1}(D)$ such that for all $\varphi \in \mathcal{S}(D)$, the map $t \mapsto (\langle u(t),\varphi\rangle_0, \langle v(t),\varphi\rangle_{-1,1})$ is $\mathbb{P}-a.s.$ continuous on $\mathbb{R}_+$ and satisfies, for all $t \in \mathbb{R}_+$,

$$\begin{cases} \langle u(t),\varphi\rangle_0 = \langle u_0,\varphi\rangle_0 + \displaystyle\int_0^t ds\ \langle v(s),\varphi\rangle_{-1,1}, \\ \\ \langle v(t),\varphi\rangle_{-1,1} = \langle v_0,\varphi\rangle_{-1,1} + \displaystyle\int_0^t ds\ (-2a\ \langle v(s),\varphi\rangle_{-1,1} - b\ \langle u(s),\varphi\rangle_0 + \langle u(s),\Delta\varphi\rangle_0) + F_t^D(\varphi). \end{cases} \tag{2.4}$$

Moreover, we say that the weak solution of equation (2.1) is *unique* if for any two solutions $(u^{(1)},v^{(1)})$ and $(u^{(2)},v^{(2)})$,

$$u^{(1)}(t) = u^{(2)}(t) \quad \text{and} \quad v^{(1)}(t) = v^{(2)}(t),$$

for all $t \in \mathbb{R}_+$, $\mathbb{P}-a.s.$

Remark 2.3.1. In the following, we will often be loosely speaking of u, instead of (u,v), for the solution of equation (2.4).

Remark 2.3.2. A solution u of (2.4) is termed a “weak” solution of equation (2.1), because it takes its values in $L^2(D)$, and therefore neither Δu nor $\frac{\partial u}{\partial \nu}\big|_{\partial D}$ are defined. A stronger way of defining a solution (u,v) of equation (2.1) is to impose that it takes its values in $H^1(D) \oplus L^2(D)$. Nevertheless, we will see in the next chapter that there never exists such a solution when the noise is a boundary noise (see also Remark 2.5.6 in the present chapter). Furthermore, we will see in Section 3.2 that for this kind of noise, equation (2.4) can be reinterpreted as the weak formulation of an equation which is different from the original equation (2.1).

2.4 Green kernel decomposed in eigenmodes

Let $n \in \mathbb{N}$ and $G_n : \mathbb{R} \to \mathbb{R}$ be the function solution of

$$G_n''(t) + 2a\ G_n'(t) + (b + \lambda_n)\ G_n(t) = 0, \qquad G_n(0) = 0, \qquad G_n'(0) = 1. \tag{2.5}$$

Thus, G_n is given by

$$G_n(t) = \begin{cases} e^{-at}\ \dfrac{\sin\left(t\sqrt{\lambda_n + b - a^2}\right)}{\sqrt{\lambda_n + b - a^2}}, & \text{if} \quad \lambda_n > a^2 - b, \\ e^{-at}\ t, & \text{if} \quad a^2 - b \geq 0 \quad \text{and} \quad \lambda_n = a^2 - b, \\ e^{-at}\ \dfrac{\sinh\left(t\sqrt{a^2 - b - \lambda_n}\right)}{\sqrt{a^2 - b - \lambda_n}}, & \text{if} \quad a^2 - b > 0 \quad \text{and} \quad \lambda_n < a^2 - b. \end{cases} \tag{2.6}$$

Note that the first of these three expressions contains actually the other two, since we have $\lim_{u\to 0} \frac{\sin(u)}{u} = 1$ and $\sin(iu) = i\sinh(u)$. The following estimates are easy to obtain: for all $T > 0$, there exists $C(T) > 0$ such that

$$|G_n(t)| \leq C(T), \qquad \forall t \in [0,T],\ n \in \mathbb{N}, \tag{2.7}$$

and

$$|G_n'(t)| \leq C(T), \qquad \forall t \in [0,T],\ n \in \mathbb{N}. \tag{2.8}$$

Moreover, we have the following lemma.

Lemma 2.4.1. *For all $t > 0$, there exist $C_-(t)$, $C_+(t) > 0$ and $n_0(t) \in \mathbb{N}$ such that*

$$\frac{C_-(t)}{1+\lambda_n} \leq \int_0^t ds\ G_n(s)^2 \leq \frac{C_+(t)}{1+\lambda_n},$$

for all $n \geq n_0(t)$.

Proof. Let $n_0(t) \in \mathbb{N}$ be such that $\lambda_{n_0(t)} \geq 2\,(a^2-b) + (1 \vee \frac{1}{t^2})$ and $n \geq n_0(t)$. Then

$$\int_0^t ds\ G_n(s)^2 = \int_0^t ds\ e^{-2as}\,\frac{\sin^2\left(s\sqrt{\lambda_n + b - a^2}\right)}{\lambda_n + b - a^2}. \tag{2.9}$$

Let us compute the upper bound first. Using the fact that $e^{-2as} \leq e^{2a^-t}$, where a^- denotes the negative part of a, we obtain

$$\int_0^t ds\ G_n(s)^2 \leq \frac{t\ e^{2a^-t}}{\lambda_n + b - a^2}$$

Since $\lambda_n \geq 2\,(a^2-b)+1$, we also have

$$\lambda_n + b - a^2 \geq \lambda_n + \frac{1-\lambda_n}{2} \geq \frac{1+\lambda_n}{2}, \tag{2.10}$$

so we obtain

$$\int_0^t ds\ G_n(s)^2 \leq \frac{t\ e^{2a^-t}}{2\,(1+\lambda_n)}.$$

The lower bound is obtained as follows. Denote by a^+ the positive part of a. Formula (2.9) then implies

$$\int_0^t ds\ G_n(s)^2 \geq \frac{e^{-2a^+t}}{\lambda_n + b - a^2}\,\frac{t}{2}\left(1 - \frac{\sin\left(2t\sqrt{\lambda_n + b - a^2}\right)}{2t\sqrt{\lambda_n + b - a^2}}\right).$$

Since $t\sqrt{\lambda_n + b - a^2} \geq 1$, we have

$$\left(1 - \frac{\sin\left(2t\sqrt{\lambda_n + b - a^2}\right)}{2t\sqrt{\lambda_n + b - a^2}}\right) \geq \frac{1}{2},$$

and since

$$\begin{aligned} \lambda_n + b - a^2 &\leq \begin{cases} 1+\lambda_n, & \text{if} \quad b - a^2 \leq 1, \\ (b-a^2)\,(1+\lambda_n), & \text{if} \quad b - a^2 \geq 1, \end{cases} \\ &\leq (1 \vee (b-a^2))\,(1+\lambda_n), \end{aligned} \tag{2.11}$$

we obtain

$$\int_0^t ds\ G_n(s)^2 \geq \frac{t\ e^{-2a^+t}}{4\,(1 \vee (b-a^2))\,(1+\lambda_n)}.$$

This completes the proof. □

Let us also define, for $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $H_n(t) = G'_n(t) + 2a\, G_n(t)$. We easily see that H_n satisfies

$$H''_n(t) + 2a\, H'_n(t) + (b + \lambda_n)\, H_n(t) = 0, \qquad H_n(0) = 1, \qquad H'_n(0) = 0. \tag{2.12}$$

The equation follows directly from the definition of H_n and equation (2.5). In order to check the initial conditions, let us compute

$$H_n(t) = e^{-at} \cos\left(t\sqrt{\lambda_n + b - a^2}\right) + a\, e^{-at}\, \frac{\sin\left(t\sqrt{\lambda_n + b - a^2}\right)}{\sqrt{\lambda_n + b - a^2}}, \tag{2.13}$$

therefore $H_n(0) = 1$, and

$$H'_n(t) = -e^{-at}\sqrt{\lambda_n + b - a^2}\, \sin\left(t\sqrt{\lambda_n + b - a^2}\right) - a^2\, e^{-at}\, \frac{\sin\left(t\sqrt{\lambda_n + b - a^2}\right)}{\sqrt{\lambda_n + b - a^2}},$$

therefore $H'_n(0) = 0$. Moreover, the following estimates are easy to obtain: for all $T > 0$, there exists $C(T) > 0$ such that

$$|H_n(t)| \leq C(T), \qquad \forall t \in [0, T],\ n \in \mathbb{N}, \tag{2.14}$$

and

$$|H'_n(t)| \leq C(T)\, \sqrt{1 + \lambda_n}, \qquad \forall t \in [0, T],\ n \in \mathbb{N}. \tag{2.15}$$

2.5 Existence and uniqueness of the solution

What we will show is that there exists a unique weak solution to equation (2.1) under the following assumption. Let us denote $\gamma_{n,m} = \Gamma_D(e_n, e_m)$ for $n, m \in \mathbb{N}$, where the e_n are the elements of the Hilbertian basis of Theorem 2.1.1, and $\gamma_n = \gamma_{n,n}$ for $n \in \mathbb{N}$.

Assumption H_0**.**
(i) Γ_D is continuous with respect to the H^1-norm, that is, there exists $C > 0$ such that

$$\Gamma_D(\varphi, \varphi) \leq C\, \|\varphi\|_1^2, \qquad \forall \varphi \in \mathcal{S}(D).$$

(ii) The following condition is satisfied:

$$\sum_{n \in \mathbb{N}} \frac{\gamma_n}{1 + \lambda_n} < \infty.$$

Remark 2.5.1. Part (i) of the above assumption implies that there exists $Q_D \in \mathcal{L}(H^1(D))$, the space of linear continuous operators on $H^1(D)$, such that

$$\Gamma_D(\varphi, \psi) = \langle \varphi, Q_D \psi \rangle_1, \qquad \forall \varphi, \psi \in \mathcal{S}(D).$$

Assuming that this is true, part (ii) of assumption H_0 implies moreover that $Q_D \in \mathcal{L}^1(H^1(D))$, the space of trace-class linear operators on $H^1(D)$, that is,

$$\sum_{n \in \mathbb{N}} \langle f_n, Q_D f_n \rangle_1 < \infty,$$

where $\{f_n,\ n \in \mathbb{N}\}$ is any Hilbertian basis of $H^1(D)$.

Furthermore, note that by estimate (2.2), we could make part (ii) of Assumption H_0 more precise, namely

$$\sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+n^{2/d}} < \infty,$$

but this will not be needed in the study of the following chapter, since we will consider in there a different way of ordering the eigenvalues λ_n.

We will also need the following stochastic Fubini theorem.

Theorem 2.5.2. *If $W = \{W_s,\ s \in \mathbb{R}_+\}$ is a standard Brownian motion, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ is continuous and $t \in \mathbb{R}_+$, then $\mathbb{P} - a.s.$,*

$$\int_0^t ds \int_0^s dW_r\, g(r,s) = \int_0^t dW_r \int_r^t ds\, g(r,s).$$

Proof. We use here the fact that the two square-integrable random variables

$$X_1 = \int_0^t ds \int_0^s dW_r\, g(r,s) \quad \text{and} \quad X_2 = \int_0^t dW_r \int_r^t ds\, g(r,s)$$

are equal $\mathbb{P} - a.s.$ if and only if $\mathbb{E}(X_1^2) = \mathbb{E}(X_1X_2) = \mathbb{E}(X_2^2)$ (that is, the variances and the covariance of the two terms are all equal). Let us then compute

$$\begin{aligned}
\mathbb{E}(X_1^2) &= \int_0^t ds \int_0^t dp\, \mathbb{E}\Big(\int_0^s dW_r\, g(r,s) \int_0^p dW_q\, g(q,s)\Big)\\
&= \int_0^t ds \int_0^t dr \int_0^{s\wedge p} dr\, g(r,s)\, g(r,p),\\
\mathbb{E}(X_2^2) &= \mathbb{E}\bigg(\Big(\int_0^t dW_r \int_r^t ds\, g(r,s)\Big)^2\bigg) = \int_0^t dr\, \Big(\int_r^t ds\, g(r,s)\Big)^2\\
&= \int_0^t dr \int_r^t ds \int_r^t dp\, g(r,s)\, g(r,p),
\end{aligned}$$

and finally,

$$\begin{aligned}
\mathbb{E}(X_1X_2) &= \int_0^t ds\, \mathbb{E}\Big(\int_0^s dW_r\, g(r,s) \int_0^t dW_r \int_r^t dp\, g(r,p)\Big)\\
&= \int_0^t ds \int_0^s dr \int_r^t dp\, g(r,s)\, g(r,p).
\end{aligned}$$

Since these three integrals are integrals of the same function on the same domain:

$$\left\{ (s,r,p) \in [0,t]^3 \,\middle|\, r \le s \quad \text{and} \quad r \le p \right\},$$

we obtain the desired result. □

Let us now state the two main theorems of this section.

Theorem 2.5.3. *Let $(u_0, v_0) \in L^2(D) \oplus H^{-1}(D)$. Under Assumption H_0, the process $(u,v) = \{(u(t), v(t)),\ t \in \mathbb{R}_+\}$ with values in $L^2(D) \oplus H^{-1}(D)$ defined by*

$$u(t) = \sum_{n\in\mathbb{N}} (u_n^0(t) + p_n(t))\ e_n, \quad \text{and} \quad v(t) = \sum_{n\in\mathbb{N}} (v_n^0(t) + q_n(t))\ e_n, \tag{2.16}$$

where

$$\begin{cases} u_n^0(t) = H_n(t)\ \langle u_0, e_n\rangle_0 + G_n(t)\ \langle v_0, e_n\rangle_{-1,1}, \quad p_n(t) = \displaystyle\int_0^t dF_s^D(e_n)\ G_n(t-s), \\ v_n^0(t) = H_n'(t)\ \langle u_0, e_n\rangle_0 + G_n'(t)\ \langle v_0, e_n\rangle_{-1,1}, \quad q_n(t) = \displaystyle\int_0^t dF_s^D(e_n)\ G_n'(t-s), \end{cases}$$

admits a modification $(\tilde u, \tilde v)$ which is the unique weak solution of equation (2.1). Moreover, $\mathbb{E}(\|\tilde u(t)\|_0^2) < \infty$ and $\mathbb{E}(\|\tilde v(t)\|_{-1}^2) < \infty$, for all $t \in \mathbb{R}_+$.

Theorem 2.5.4. *Let $(u_0, v_0) \in L^2(D) \oplus H^{-1}(D)$. If there exists a weak solution (u, v) to equation (2.1) such that $\mathbb{E}(\|u(t_0)\|_0^2) < \infty$, for some $t_0 > 0$, then part (ii) of Assumption H_0 is satisfied.*

Remark 2.5.5. Note that these two results belong to the general theory developed by G. Da Prato and J. Zabczyk in [18], but since the case that we consider here is a simple one, we rewrite the proofs in this simple case for clarity and completeness.

Proof of Theorem 2.5.3. Let us first show existence. The deterministic process (u^0, v^0) defined by

$$u^0(t) = \sum_{n\in\mathbb{N}} u_n^0(t)\ e_n \quad \text{and} \quad v^0(t) = \sum_{n\in\mathbb{N}} v_n^0(t)\ e_n,$$

takes its values in $L^2(D) \oplus H^{-1}(D)$ by estimates (2.7), (2.8), (2.14) and (2.15). By a direct calculation using equations (2.5) and (2.12), we see that (u_n^0, v_n^0) satisfies, for a fixed $n \in \mathbb{N}$,

$$\begin{cases} u_n^0(t) = \langle u_0, e_n\rangle_0 + \displaystyle\int_0^t ds\ v_n^0(s), \\ v_n^0(t) = \langle v_0, e_n\rangle_{-1,1} - \displaystyle\int_0^t ds\ (2a\ v_n^0(s) + (b + \lambda_n)\ u_n^0(s)). \end{cases}$$

Multiplying this equation by $\langle e_n, \varphi\rangle_0$ and summing over $n \in \mathbb{N}$ gives then the following equation for (u^0, v^0), after some permutations of sums and integrals:

$$\begin{cases} \langle u^0(t), \varphi\rangle_0 = \langle u_0, \varphi\rangle_0 + \displaystyle\int_0^t ds\ \langle v^0(s), \varphi\rangle_{-1,1}, \\ \langle v^0(t), \varphi\rangle_{-1,1} = \langle v_0, \varphi\rangle_{-1,1} + \displaystyle\int_0^t ds\ (-2a\ \langle v^0(s), \varphi\rangle_{-1,1} - b\ \langle u^0(s), \varphi\rangle_0 + \langle u^0(s), \Delta\varphi\rangle_0). \end{cases} \tag{2.17}$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(D)$.

On the other hand, integrating equation (2.5) in t, then with respect to the Brownian motion $F^D(e_n)$ gives

$$\begin{cases} \displaystyle\int_0^t dF_s^D(e_n)\; G_n(t-s) = \int_0^t dF_s^D(e_n) \int_s^t dr\; G_n'(r-s), \\ \displaystyle\int_0^t dF_s^D(e_n)\; G_n'(t-s) = F_t^D(e_n) - \int_0^t dF_s^D(e_n) \int_s^t dr\; (2a\; G_n'(r-s) + (b+\lambda_n)\; G_n(r-s)). \end{cases}$$

Applying the stochastic Fubini theorem 2.5.2 to the integral terms, we obtain that the process (p_n, q_n) defined in the theorem satisfies

$$\begin{cases} p_n(t) = \displaystyle\int_0^t ds\; q_n(s), \\ q_n(t) = F_t^D(e_n) - \displaystyle\int_0^t ds\; (2a\; q_n(s) + (b+\lambda_n)\; p_n(s)). \end{cases} \tag{2.18}$$

Let us now define the process (p, q) by

$$p(t) = \sum_{n\in\mathbb{N}} p_n(t)\; e_n \quad \text{and} \quad q(t) = \sum_{n\in\mathbb{N}} q_n(t)\; e_n.$$

We first check that $\mathbb{E}(\|p(t)\|_0^2) < \infty$, for all $t \in \mathbb{R}_+$:

$$\mathbb{E}\left(\|p(t)\|_0^2\right) = \sum_{n\in\mathbb{N}} \mathbb{E}(p_n(t)^2) \leq \sum_{n<n_0(t)} \gamma_n \int_0^t ds\; G_n(t-s)^2 + C_+(t) \sum_{n\geq n_0(t)} \frac{\gamma_n}{1+\lambda_n} < \infty,$$

by the upper bound in Lemma 2.4.1 and part (ii) of Assumption H_0. Moreover, let us check that $\mathbb{E}(\|q(t)\|_{-1}^2) < \infty$, for all $t \in \mathbb{R}_+$:

$$\mathbb{E}\left(\|q(t)\|_{-1}^2\right) = \sum_{n\in\mathbb{N}} \frac{\mathbb{E}(q_n(t)^2)}{1+\lambda_n} \leq C(t) \sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+\lambda_n} < \infty,$$

by estimate (2.8) and part (ii) of Assumption H_0. We then have, using the fact that the Laplacian is symmetric on $\mathcal{S}(D)$,

$$\begin{aligned} \langle p(t), \Delta\varphi\rangle_0 &= \sum_{n\in\mathbb{N}} p_n(t)\; \langle e_n, \Delta\varphi\rangle_0 = \sum_{n\in\mathbb{N}} p_n(t)\; \langle \Delta e_n, \varphi\rangle_0, \\ &= -\sum_{n\in\mathbb{N}} \lambda_n\; p_n(t)\; \langle e_n, \varphi\rangle_0, \end{aligned}$$

by Theorem 2.1.1. Multiplying equation (2.18) by $\langle e_n, \varphi\rangle_0$ and summing over $n \in \mathbb{N}$ gives then the following equation for (p, q), after some permutations of sums and integrals:

$$\begin{cases} \langle p(t), \varphi\rangle_0 = \displaystyle\int_0^t ds\; \langle q(s), \varphi\rangle_{-1,1}, \\ \langle q(t), \varphi\rangle_{-1,1} = F_t^D(\varphi) + \displaystyle\int_0^t ds\; (-2a\; \langle q(s), \varphi\rangle_{-1,1} - b\; \langle p(s), \varphi\rangle_0 + \langle p(s), \Delta\varphi\rangle_0), \end{cases} \tag{2.19}$$

$\mathbb{P}-a.s$, for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(D)$, where we have used the fact that

$$\sum_{n\in\mathbb{N}} F_r^D(e_n)\,\langle e_n,\varphi\rangle_0 = F_r^D(\varphi) \quad \mathbb{P}-a.s,\ \forall t\in\mathbb{R}_+,\ \varphi\in\mathcal{S}(D),$$

by part (i) of Assumption H_0. Combining finally equations (2.17) and (2.19), and using the Kolmogorov continuity theorem (see [29, Thm 2.8]) shows that the process $(u,v) = \{(u(t),v(t)), t \in \mathbb{R}_+\}$ defined by (2.16) admits a modification $(\tilde{u},\tilde{v})$ such that for all $\varphi \in \mathcal{S}(D)$, the map $t \mapsto (\langle \tilde{u}(t),\varphi\rangle_0, \langle \tilde{v}(t),\varphi\rangle_{-1,1})$ is $\mathbb{P}-a.s.$ continuous and solves equation (2.4).

In order to prove uniqueness, let $(u^{(1)},v^{(1)})$ and $(u^{(2)},v^{(2)})$ be two solutions of equation (2.4) and define $(\bar{u},\bar{v}) = (u^{(1)}-u^{(2)}, v^{(1)}-v^{(2)})$. For all $\varphi \in \mathcal{S}(D)$, there exists a $\mathbb{P}$-null set such that outside this set, the following equation satisfied for all $t \in \mathbb{R}_+$:

$$\begin{cases} \langle \bar{u}(t),\varphi\rangle_0 = \displaystyle\int_0^t ds\ \langle \bar{v}(s),\varphi\rangle_{-1,1}, \\ \langle \bar{v}(t),\varphi\rangle_{-1,1} = \displaystyle\int_0^t ds\ (-2a\ \langle \bar{v}(s),\varphi\rangle_{-1,1}) - b\ \langle \bar{u}(s),\varphi\rangle_0 + \langle \bar{u}(s),\Delta\varphi\rangle_0). \end{cases}$$

Fix now $n \in \mathbb{N}$ and define $\bar{u}_n(t) = \langle \bar{u}(t), e_n\rangle_0$ and $\bar{v}_n(t) = \langle \bar{v}(t), e_n\rangle_{-1,1}$, for $t \in \mathbb{R}_+$. Replacing φ by e_n in the preceding equation and using the symmetry of the Laplacian on $\mathcal{S}(D)$, we obtain that for all $n \in \mathbb{N}$, $(\bar{u}_n, \bar{v}_n)$ satisfies, outside a $\mathbb{P}$-null set and for all $t \in \mathbb{R}_+$,

$$\begin{cases} \bar{u}_n(t) = \displaystyle\int_0^t ds\ \bar{v}_n(s), \\ \bar{v}_n(t) = -\displaystyle\int_0^t ds\ (2a\ \bar{v}_n(s) + (b+\lambda_n)\ \bar{u}_n(s)). \end{cases}$$

Therefore, for all $n \in \mathbb{N}$, we have that $\bar{u}_n(t) = \bar{v}_n(t) = 0$ for all $t \in \mathbb{R}_+$, outside a $\mathbb{P}$-null set. Since $\mathbb{N}$ is countable and $\bar{u},\bar{v}$ are entirely determined by their components $\bar{u}_n, \bar{v}_n$, the conclusion follows. □

Proof of Theorem 2.5.4. Let (u,v) be a solution of equation (2.4) and let $t_0 > 0$ be such that $\mathbb{E}(\|u(t_0)\|_0^2) < \infty$. Let us then replace φ by e_n in equation (2.4) and denote $u_n(t) = \langle u(t), e_n\rangle_0$ and $v_n(t) = \langle v(t), e_n\rangle_{-1,1}$. By calculations similar to those of the proof of the preceding theorem, we obtain that

$$u_n(t) = u_n^0(t) + p_n(t) \quad \text{and} \quad v_n(t) = v_n^0(t) + q_n(t),$$

where

$$\begin{cases} u_n^0(t) = H_n(t)\ \langle u_0, e_n\rangle_0 + G_n(t)\ \langle v_0, e_n\rangle_{-1,1}, \quad p_n(t) = \displaystyle\int_0^t dF_s^D(e_n)\ G_n(t-s), \\ v_n^0(t) = H_n'(t)\ \langle u_0, e_n\rangle_0 + G_n'(t)\ \langle v_0, e_n\rangle_{-1,1}, \quad q_n(t) = \displaystyle\int_0^t dF_s^D(e_n)\ G_n'(t-s), \end{cases}$$

Since, for the same reasons as before, the process (u^0, v^0) defined by

$$u^0(t) = \sum_{n\in\mathbb{N}} u_n^0(t)\ e_n \quad \text{and} \quad v^0(t) = \sum_{n\in\mathbb{N}} v_n^0(t)\ e_n,$$

belongs to $L^2(D) \oplus H^{-1}(D)$ for all $t \in \mathbb{R}_+$, the assumption made on u then implies that if $p(t) = \sum_{n\in\mathbb{N}} p_n(t)\ e_n$, then

$$\mathbb{E}(\|p(t_0)\|_0^2) < \infty.$$

But a direct calculation shows that

$$\mathbb{E}(\|p(t_0)\|_0^2) = \sum_{n\in\mathbb{N}} \gamma_n \int_0^{t_0} ds\ G_n(t-s)^2 \geq C_-(t_0) \sum_{n\geq n_0(t_0)} \frac{\gamma_n}{1+\lambda_n},$$

by the lower bound in Lemma 2.4.1, so part (ii) of Assumption H_0 must be satisfied, and this completes the proof. □

Remark 2.5.6. Performing the same kind of analysis as above, we could see that if there exists a solution (u, v) to equation (2.4) with values in $H^1(D) \oplus L^2(D)$, then the following condition (stronger than part (ii) of Assumption H_0) must be satisfied:

$$\sum_{n\in\mathbb{N}} \gamma_n < \infty. \tag{2.20}$$

Nevertheless, this latter condition is never satisfied in the case of a boundary noise, as we will see in the next chapter, so there does not exist a solution with values in $H^1(D) \oplus L^2(D)$ in this case.

2.6 Heat equation

If, instead of the hyperbolic equation considered above, we rather consider the following parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \frac{1}{2}\,\Delta u(t,x) = \dot{F}^D(t,x), & (t,x) \in \mathbb{R}_+ \times D, \\[2ex] \dfrac{\partial u}{\partial \nu}(t,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \partial D, \\[2ex] u(0,x) = u_0(x), \qquad x \in D, \end{cases} \tag{2.21}$$

we can then reproduce the entire analysis of the preceding sections. The only difference will consist in the fact that the weak formulation is simpler to express (we only have one process u taking its values in $L^2(D)$) and that the G_n are solutions of

$$G_n'(t) + \frac{\lambda_n}{2}\,G_n(t) = 0, \qquad G_n(0) = 1. \tag{2.22}$$

They are therefore given by

$$G_n(t) = \exp(-\frac{\lambda_n t}{2}). \tag{2.23}$$

The analysis is similar to that of the hyperbolic case because these G_n also satisfy Lemma 2.4.1, so Theorems 2.5.3 and 2.5.4 (adapted to the present situation) remain valid in the case of the heat equation.

Nevertheless, qualitative differences appear between the behavior of the solution of the hyperbolic and the heat equation in the case of a boundary noise. These will be explained in the next chapter.

Chapter 3

Noise on a sphere

Let d be a natural number greater than one, $B(0,1)$ the centered unit ball in $\mathbb{R}^d$ and $\partial B(0,1) = S^{d-1}$ the centered unit sphere embedded in $\mathbb{R}^d$. In this chapter, we would like to study the existence of a weak solution to the hyperbolic equation (2.1) (in the sense defined in (2.4)), in the specific case where the domain $D = B(0,1)$ and the noise considered is concentrated on the sphere S^{d-1}.

3.1 Eigenvalues and eigenfunctions of the Laplacian in $B(0,1)$

Let us first define the following Bessel functions for $l \in \mathbb{N}$ and $d \geq 2$:

$$J_l(d,r) = \Gamma\left(\frac{d}{2}\right) \left(\frac{r}{2}\right)^{\frac{2-d}{2}} J_{l+\frac{d-2}{2}}(r), \qquad r > 0, \tag{3.1}$$

where Γ is the Euler Gamma function defined by

$$\Gamma(\nu) = \int_0^\infty dt\, t^{\nu-1}\, e^{-t}, \qquad \nu > 0, \tag{3.2}$$

and J_ν is the regular Bessel function of order ν of the first kind (see Appendix B for a definition). In the following, we will also need the expression of the derivative of $J_l(d,\cdot)$ in r when $d > 2$:

$$J_l'(d,r) = \Gamma\left(\frac{d}{2}\right) \left(\frac{r}{2}\right)^{\frac{2-d}{2}} \left(J'_{l+\frac{d-2}{2}}(r) - \frac{d-2}{2r}\, J_{l+\frac{d-2}{2}}(r)\right). \tag{3.3}$$

Let us now describe precisely the solutions of the following eigenvalue problem:

$$\Delta\varphi + \lambda\varphi = 0 \text{ in } B(0,1) \quad \text{and} \quad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial B(0,1)} = 0,$$

which exist by the spectral theorem 2.1.1. By standard theory (see [41, §22]), they are of the form $\varphi(x) = f(r)\, Y(\theta)$, where $r = |x|$ and θ is a vector of dimension $d-1$ representing the angular part of x. Y is solution of the following eigenvalue problem:

$$\Delta_\theta Y(\theta) + \Lambda\, Y(\theta) = 0,$$

where Δ_θ denotes the Laplace-Beltrami operator on S^{d-1}. The solutions of this problem are well known and given by

$$\{\Lambda_l,\ Y_l^m,\ l \in \mathbb{N},\ 1 \leq m \leq N(d,l)\},$$

where $\Lambda_l = l(l+d-2)$, $\{Y_l^m,\ 1 \le m \le N(d,l)\}$ is the list of generalized spherical harmonics of order l on S^{d-1} and $N(d,l)$ is the number of these harmonics (see [41, §15]). Note that when $d=2$, $N(2,l)=2$ and $Y_l^{\pm}(\theta) = \exp(\pm il\theta)$; when $d=3$, $N(3,l) = 2l+1$ and the Y_l^m are the standard spherical harmonics on S^2.

For a fixed $l \in \mathbb{N}$, f is now solution of the following eigenvalue problem

$$f''(r) + \frac{d-1}{r}\, f'(r) + \left(\lambda - \frac{l(l+d-2)}{r^2}\right)\, f(r) = 0, \qquad f'(1) = 0,$$

The solutions of this problem are also well known (see [41, §22]) and given by

$$\{\lambda_{kl},\ f_{kl},\ k \in \mathbb{N}\}$$

where $\lambda_{kl} = \mu_{kl}^2$, with $\{\mu_{kl},\ k \in \mathbb{N}\}$ the ascending list of zeros, for a fixed $l \in \mathbb{N}$, of the derivative of the Bessel function $J_l(d,\cdot)$ defined by (3.1), and f_{kl} is the function defined, for fixed $k,l \in \mathbb{N}$, by

$$f_{kl}(r) = \frac{J_l(d, \mu_{kl}\, r)}{\sqrt{\int_0^1 dq\; q^{d-1}\; J_l(d, \mu_{kl}\, q)^2}}.$$

This gives finally the following set of eigenvalues and eigenfunctions of the Laplacian in $B(0,1)$:

$$\{\lambda_{kl},\ e_{klm} = f_{kl} \otimes Y_l^m,\ k,l \in \mathbb{N},\ 1 \le m \le N(d,l)\},$$

the above "tensor product" being understood as $e_{klm}(x) = f_{kl}(r)\; Y_l^m(\theta)$.

Note that these eigenfunctions are normalized in $L^2(B(0,1))$, that is,

$$\int_{B(0,1)} dx\; |e_{klm}(x)|^2 = 1, \qquad \forall k,l \in \mathbb{N},\ 1 \le m \le N(d,l),$$

and let us mention the two following facts, which will be used in the next section.

Lemma 3.1.1. *For all $k,l \in \mathbb{N}$,*

$$f_{kl}(1)^2 = \frac{2\,\lambda_{kl}}{\lambda_{kl} - l^2 - l(d-2)}.$$

Proof. Let us first compute the normalizing factor in $f_{kl}(1)^2$, using formula (6.52) p.101 in [7]:

$$\int_0^1 dq\; q\; J_l(\alpha\, q)^2 = \frac{1}{2}\left(J_l'(\alpha)^2 + \left(1 - \frac{l^2}{\alpha^2}\right)\; J_l(\alpha)^2\right), \qquad \alpha > 0.$$

When $d = 2$, we therefore have

$$\int_0^1 dq\; q\; J_l(\mu_{kl}\, q)^2 = \frac{1}{2}\left(1 - \frac{l^2}{\lambda_{kl}}\right)\; J_l(\mu_{kl})^2$$

since $J_l'(\mu_{kl}) = 0$. From this, we see immediately that

$$f_{kl}(1)^2 = \frac{J_l(\mu_{kl})^2}{\int_0^1 dq\; q^{d-1}\; J_l(\mu_{kl}\, q)^2} = \frac{2\,\lambda_{kl}}{\lambda_{kl} - l^2}.$$

When $d > 2$, we have, by definition of $J_l(d,\cdot)$ and the above formula,

$$\int_0^1 dq\, q^{d-1}\, J_l(d,\mu_{kl}\, q)^2 = \Gamma\left(\frac{d}{2}\right)^2 \left(\frac{\mu_{kl}}{2}\right)^{2-d} \int_0^1 dq\, q\, J_{l+\frac{d-2}{2}}(\mu_{kl}\, q)^2$$
$$= \frac{1}{2}\,\Gamma\left(\frac{d}{2}\right)^2 \left(\frac{\mu_{kl}}{2}\right)^{2-d} \left(J'_{l+\frac{d-2}{2}}(\mu_{kl})^2 + \left(1 - \frac{(l+\frac{d-2}{2})^2}{\lambda_{kl}}\right) J_{l+\frac{d-2}{2}}(\mu_{kl})^2\right).$$

But using (3.3) and the fact that $J_l'(d,\mu_{kl}) = 0$, we have

$$J'_{l+\frac{d-2}{2}}(\mu_{kl}) = \frac{d-2}{2\,\mu_{kl}}\, J_{l+\frac{d-2}{2}}(\mu_{kl}),$$

so

$$\int_0^1 dq\, q^{d-1}\, J_l(d,\mu_{kl}\, q)^2 = \frac{1}{2}\left(\frac{\lambda_{kl} - l^2 - l(d-2)}{\lambda_{kl}}\right) J_l(d,\mu_{kl})^2,$$

therefore,

$$f_{kl}(1)^2 = \frac{2\,\lambda_{kl}}{\lambda_{kl} - l^2 - l(d-2)},$$

and this completes the proof. □

Lemma 3.1.2. *For all* $k, l \in \mathbb{N}$,

$$l + \frac{d-2}{2} + \pi(k-2) \leq \mu_{kl} \leq \frac{\pi}{2}(l + \frac{d-2}{2}) + \pi(k+2).$$

Proof. Let us denote by $\{\mu^0_{k\nu}, k \in \mathbb{N}\}$ the list of zeros of the standard Bessel function J_ν (the first zero being therefore indexed by $k = 0$); by Theorem 1 and Lemma 2 in [9], we have

$$\nu + \pi(k-1) \leq \mu^0_{k\nu} \leq \frac{\pi}{2}\nu + \pi(k+1).$$

Since $J_l(d,\cdot)$ is proportional to $J_{l+\frac{d-2}{2}}(\cdot)$ and by the interlacing property of the zeros of Bessel functions and their derivatives (that is, if ν_{kl} are the zeros of $J_l(d,\cdot)$, then $\nu_{k-1,l} < \mu_{kl} < \nu_{k+1,l}$), we have

$$\mu^0_{k-1,l+\frac{d-2}{2}} \leq \mu_{kl} \leq \mu^0_{k+1,l+\frac{d-2}{2}},$$

so we obtain that

$$l + \frac{d-2}{2} + \pi(k-2) \leq \mu_{kl} \leq \frac{\pi}{2}(l + \frac{d-2}{2}) + \pi(k+2),$$

which completes the proof. □

Let us end this section with the following comment. The generalized spherical harmonics

$$\{Y_l^m,\ l \in \mathbb{N},\ 1 \leq m \leq N(d,l)\}$$

form a Hilbertian basis of $L^2(S^{d-1})$ and are the eigenfunctions of the Laplace-Beltrami operator Δ_θ on S^{d-1}, with corresponding eigenvalues $l(l+d-2)$, as mentioned before. Let us now define $H^{\frac{1}{2}}(S^{d-1})$ as the Sobolev space of order $\frac{1}{2}$ on S^{d-1} (that is, the domain of $(I-\Delta_\theta)^{\frac{1}{4}}$ in $L^2(S^{d-1})$, see [59, p. 255]). By the spectral decomposition of Δ_θ in the spherical harmonics, we therefore obtain the following characterization of $H^{\frac{1}{2}}(S^{d-1})$:

$$H^{\frac{1}{2}}(S^{d-1}) = \left\{ v = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} c_{lm}\, Y_l^m \;\middle|\; \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} (1+l)\,|c_{lm}|^2 < \infty \right\},$$

and we equip this space with the norm

$$\|v\|_{\frac{1}{2}}^2 = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} (1+l)\,|c_{lm}|^2.$$

Interesting to us is the following relation between $H^1(B(0,1))$ and $H^{\frac{1}{2}}(S^{d-1})$ (see [2, Thm 7.53]):

$$H^{\frac{1}{2}}(S^{d-1}) = \left\{ v = \gamma_0 u \;\middle|\; u \in H^1(B(0,1)) \right\},$$

where γ_0 is the trace operator of $H^1(B(0,1))$ on $H^{\frac{1}{2}}(S^{d-1})$, defined by

$$\gamma_0 \varphi = \varphi\big|_{S^{d-1}}, \qquad \forall \varphi \in C^\infty(\overline{B(0,1)}),$$

and further extended by continuity to $H^1(B(0,1))$. The operator γ_0 is continuous with respect to the norm $\|\cdot\|_1$ and there also exists an application $R_0 : H^{\frac{1}{2}}(S^{d-1}) \to H^1(B(0,1))$ which is continuous with respect to the norm $\|\cdot\|_{\frac{1}{2}}$ and such that

$$\gamma_0 R_0 v = v, \qquad \forall v \in H^{\frac{1}{2}}(S^{d-1}).$$

3.2 Covariance of the noise and Schönberg's theorem

In order to obtain a general form for the covariance of a Gaussian noise concentrated on the sphere S^{d-1}, let us first consider the case of a continuous covariance. Let $f : S^{d-1} \times S^{d-1} \to \mathbb{R}$ be a continuous function which is assumed to be symmetric and non-negative definite on S^{d-1}, that is,

$$\sum_{i,j=1}^{m} c_i\, c_j\, f(x^{(i)}, x^{(j)}) \geq 0, \qquad \forall m \geq 1,\ c_1, \ldots, c_m \in \mathbb{R},\ x^{(1)}, \ldots, x^{(m)} \in S^{d-1}.$$

This function f is then the covariance of a centered Gaussian process indexed by the elements of S^{d-1}. Let us moreover assume that the noise is *isotropic*, that is, there exists a continuous function $g : [-1,+1] \to \mathbb{R}$ such that

$$\sum_{i,j=1}^{m} c_i\, c_j\, g(x^{(i)} \cdot x^{(j)}) \geq 0, \qquad \forall m \geq 1,\ c_1, \ldots, c_m \in \mathbb{R},\ x^{(1)}, \ldots, x^{(m)} \in S^{d-1}, \tag{3.4}$$

and $f(x,y) = g(x \cdot y)$ for all $x, y \in S^{d-1}$, where $x \cdot y$ is the Euclidean scalar product of x and y in $\mathbb{R}^d$.

For $d \geq 2$, let us also define the following generalized Legendre polynomials (see [41, §2, Lemma 4])

$$P_l(d,t) = \left(-\frac{1}{2}\right)^l \frac{\Gamma(\frac{d-1}{2})}{\Gamma(l+\frac{d-1}{2})} (1-t^2)^{\frac{3-d}{2}} \left(\frac{d}{dt}\right)^l (1-t^2)^{l+\frac{d-3}{2}}, \qquad l \in \mathbb{N},\ t \in [-1,+1],$$

where Γ is the Gamma function defined in (3.2). Let us mention that these are simply the Chebychev polynomials when $d = 2$ and the standard Legendre polynomials when $d = 3$.

Schönberg's theorem (see [56, Thm 1]) states the following.

Theorem 3.2.1. *Let $g : [-1,+1] \to \mathbb{R}$ be a continuous function. Then g is non-negative definite on S^{d-1} (in the sense of (3.4)) if and only if there exists a sequence $\{a_l,\ l \in \mathbb{N}\}$ of non-negative numbers such that $\sum_{l\in\mathbb{N}} a_l < \infty$ and*

$$g(t) = \sum_{l\in\mathbb{N}} a_l\ P_l(d,t), \quad t \in [-1,+1],$$

where $P_l(d,\cdot)$ are the Legendre polynomials defined above.

This theorem, similarly to the Bochner theorem concerning non-negative definite functions on $\mathbb{R}^d$, gives us a spectral characterization of continuous non-negative definite functions on the sphere S^{d-1}.

To extend this to more general functions f, let us mention that $\Gamma_S : C^\infty(S^{d-1}) \times C^\infty(S^{d-1}) \to \mathbb{R}$ defined by

$$\Gamma_S(\varphi,\psi) = \int_{S^{d-1}} d\sigma(x) \int_{S^{d-1}} d\sigma(y)\ \varphi(x)\ g(x\cdot y)\ \psi(y), \qquad \varphi,\psi \in C^\infty(S^{d-1}),$$

(where σ is the uniform measure on the unit sphere S^{d-1}) is a semi-scalar product on $C^\infty(S^{d-1})$ under assumption (3.4). Moreover, Γ_S is *isotropic*, that is,

$$\Gamma_S(R\varphi, R\psi) = \Gamma_S(\varphi,\psi), \qquad \forall \varphi,\psi \in C^\infty(S^{d-1}),$$

for any rotation R on the sphere S^{d-1} (where $R\varphi(x) = \varphi(R^{-1}x)$ by definition).

Considering Γ_S instead of g allows us then to remove the continuity assumption on g. In view of the preceding theorem, we will therefore consider in the following that the covariance of the noise concentrated on the sphere S^{d-1} is given by

$$\Gamma_S(\varphi,\psi) = \sum_{l\in\mathbb{N}} a_l\ \Gamma_l(\varphi,\psi), \qquad \varphi,\psi \in C^\infty(S^{d-1}),$$

where

$$\Gamma_l(\varphi,\psi) = \int_{S^{d-1}} d\sigma(x) \int_{S^{d-1}} d\sigma(y)\ \varphi(x)\ P_l(d, x\cdot y)\ \psi(y),$$

and $a_l \geq 0$, but the condition $\sum_{l\in\mathbb{N}} a_l < \infty$ is replaced by

$$\sum_{l\in\mathbb{N}} \frac{a_l}{(1+l)^{r_0}} < \infty,$$

for some $r_0 > 0$. Let us check that $\Gamma_S(\varphi, \psi) < \infty$ for each $\varphi, \psi \in C^\infty(S^{d-1})$. By the Cauchy-Schwarz inequality, it is sufficient to check that $\Gamma_S(\varphi, \varphi) < \infty$ for each $\varphi \in C^\infty(S^{d-1})$. Using the fact that

$$C^\infty(S^{d-1}) \subset \bigcap_{r>0} H^r(S^{d-1}),$$

we obtain that a function φ in $C^\infty(S^{d-1})$ can be written as

$$\varphi = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} c_{lm}\, Y_l^m, \quad \text{with} \quad \sum_{l\in\mathbb{N}} (1+l)^{2r} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 < \infty, \quad \forall r > 0. \tag{3.5}$$

Using the following additivity property (see [41, §2, Thm 2]):

$$P_l(d, x\cdot y) = \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} \overline{Y_l^m(x)}\, Y_l^m(y), \tag{3.6}$$

and the orthonormality of the spherical harmonics, we can compute

$$\Gamma_l(\varphi,\varphi) = \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} \left| \int_{S^{d-1}} d\sigma(x)\, \varphi(x)\, \overline{Y_l^m(x)} \right|^2 = \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \tag{3.7}$$

Since (3.5) implies that for $r_0 > 0$, there exists $C > 0$ such that $|c_{lm}|^2 \leq \frac{C}{(1+l)^{r_0}}$ for all l, m, we obtain that

$$\Gamma_l(\varphi,\varphi) \leq \frac{C\,|S^{d-1}|}{(1+l)^{r_0}}, \qquad \forall l \in \mathbb{N}.$$

This implies finally that

$$\Gamma_S(\varphi,\varphi) = \sum_{l\in\mathbb{N}} a_l\, \Gamma_l(\varphi,\varphi) \leq C\,|S^{d-1}| \sum_{l\in\mathbb{N}} \frac{a_l}{(1+l)^{r_0}} < \infty,$$

by the assumption made above, so Γ_S is a well defined covariance on $C^\infty(S^{d-1})$.

Remark 3.2.2. It would be a nice generalization of Schönberg's theorem to prove that *every* isotropic semi-scalar product Γ on $C^\infty(S^{d-1})$, with some additional continuity property, is of the form given above. In the case of a covariance on $\mathbb{R}^d$, this extension (of the classical *Bochner* theorem) is the classical theorem of L. Schwartz (see Theorem 4.3.1).

In order to relate the particular covariance Γ_S on S^{d-1} defined above with the general covariance Γ_D which was considered in Chapter 2 and defined on the entire domain D (here equal to $B(0,1)$), we define Γ_D by

$$\Gamma_D(\varphi,\psi) = \Gamma_S\left(\varphi\big|_{S^{d-1}}, \psi\big|_{S^{d-1}}\right), \qquad \varphi,\psi \in \mathcal{S}(D),$$

where $\mathcal{S}(D)$ is the space of test functions defined in the preceding chapter.

Remark 3.2.3. Even if the particular noise defined in the present section satisfies all the requirements of the preceding chapter, it turns out, as already mentioned in Remark 2.3.2, that in this case, equation (2.4) can be reinterpreted as the weak formulation of another equation than equation (2.1). This is because the noise term $F_t^D(\varphi)$ can be formally rewritten here as

$$F_t^D(\varphi) = \int_0^t ds \int_{\partial B(0,1)} d\sigma(x)\ \dot{F}^S(s,x)\ \varphi(x), \qquad \varphi \in \mathcal{S}(D),$$

where $\dot{F}^S$ is a generalized centered Gaussian process concentrated on the sphere $\partial B(0,1) = S^{d-1}$ with covariance

$$\mathbb{E}(\dot{F}^S(t,x)\ \dot{F}^S(s,y)) = \delta_0(t-s)\ \Gamma_S(x,y).$$

Therefore, the noise term can be reinterpreted as a stochastic boundary condition and the "classical" equation corresponding to (2.4) would then be

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) + b\,u(t,x) - \Delta u(t,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times D, \\[2ex] \dfrac{\partial u}{\partial \nu}(t,x) = \dot{F}^S(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \partial D, \\[2ex] u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in D. \end{cases} \tag{3.8}$$

This interpretation of the boundary term is the one considered by R. Sowers in [61], for the heat equation.

Remark 3.2.4. Let us mention a qualitative difference between the behavior of the solution of the parabolic and the hyperbolic equations, which will be made more explicit in the following chapters concerning the equation in $\mathbb{R}^d$. For the heat equation, and because of the regularizing property of the Green kernel of this equation, the solution is always regular inside the ball $B(0,1)$ and explodes near the boundary. On the contrary, for the hyperbolic equation, the explosion, if any, occurs rather at the center of the ball, where the influence of the boundary noise is maximum for one particular time, because of the finite speed of propagation of the equation.

3.3 Explicit conditions

In the following, we give two explicit conditions on the coefficients a_l, one necessary and one sufficient, for the existence of a weak solution to equation (3.8).

In the present setting, part (ii) of Assumption H_0 of the preceding chapter can be rewritten as

$$\sum_{k,l \in \mathbb{N}} \sum_{m=1}^{N(d,l)} \frac{\gamma_{klm}}{1+\lambda_{kl}} < \infty,$$

where $\gamma_{klm} = \Gamma_D(e_{klm}, e_{klm})$. The first step consists therefore in rewriting the sum in the above expression as

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \frac{\gamma_{klm}}{1+\lambda_{kl}} = \sum_{l\in\mathbb{N}} a_l \; b_l,$$

where b_l depends on the eigenvalues and eigenfunctions of the Laplacian computed in Section 3.1. This is done in the following lemma.

Lemma 3.3.1. *For all $l \in \mathbb{N}$,*

$$b_l = |S^{d-1}| \sum_{k\in\mathbb{N}} \frac{f_{kl}(1)^2}{1+\lambda_{kl}}.$$

Proof. Let us compute

$$\begin{aligned} \gamma_{klm} &= \Gamma_D(e_{klm}, e_{klm}) = \Gamma_S(f_{kl}(1)\; Y_l^m, f_{kl}(1)\; Y_l^m) \\ &= f_{kl}(1)^2 \sum_{n\in\mathbb{N}} a_n \; \Gamma_n(Y_l^m, Y_l^m). \end{aligned}$$

By definition of Γ_n and the additivity property (3.6), we have

$$\Gamma_n(Y_l^m, Y_l^m) = \frac{|S^{d-1}|}{N(d,n)} \sum_{p=1}^{N(d,n)} \left| \int_{S^{d-1}} d\sigma(x)\; Y_l^m(x)\; \overline{Y_n^p(x)} \right|^2 = \frac{|S^{d-1}|}{N(d,n)}\; \delta_{nl}$$

since the spherical harmonics Y_l^m are orthonormalized. This implies that

$$\gamma_{klm} = f_{kl}(1)^2 \; a_l \; \frac{|S^{d-1}|}{N(d,l)},$$

therefore,

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \frac{\gamma_{klm}}{1+\lambda_{kl}} = |S^{d-1}| \sum_{l\in\mathbb{N}} a_l \left(\sum_{k\in\mathbb{N}} \frac{f_{kl}(1)^2}{1+\lambda_{kl}} \right),$$

which ends the proof. □

The second step consists in estimating the behavior of b_l in l, with the help of Lemmas 3.1.1 and 3.1.2.

Lemma 3.3.2. *There exist $C_1, C_2 > 0$ such that for sufficiently large l,*

$$C_1 \; \frac{a_l}{1+l} \le b_l \le C_2 \; \frac{a_l \; \ln(1+l)}{1+l}.$$

Proof. Using Lemma 3.1.1, we obtain that

$$\sum_{k\in\mathbb{N}} \frac{f_{kl}(1)^2}{1+\lambda_{kl}} = 2 \sum_{k\in\mathbb{N}} \frac{\lambda_{kl}}{1+\lambda_{kl}} \; \frac{1}{\lambda_{kl} - l^2 - l(d-2)}.$$

Since

$$\frac{\lambda_{kl}}{1+\lambda_{kl}} \in [\frac{1}{2}, 1]$$

for $\lambda_{kl} \geq 1$, we can remove this term from the preceding sum and study the behavior in l of

$$\sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)}.$$

Let us first prove the lower bound. Using the right-hand side of Lemma 3.1.2, we then have

$$\begin{aligned}\sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)} &\geq \sum_{k\in\mathbb{N}} \frac{1}{(\frac{\pi}{2}(l+\frac{d-2}{2}) + \pi(k+2))^2} \\ &= \frac{1}{\pi^2}\sum_{k\geq 2} \frac{1}{(\frac{l}{2}+\frac{d-2}{4}+k)^2} \geq \frac{1}{\pi^2}\int_2^\infty dx\, \frac{1}{(\frac{l}{2}+\frac{d-2}{4}+x)^2} \\ &= \frac{1}{\pi^2}\frac{1}{\frac{l}{2}+\frac{d-2}{4}+2} \geq C_1\,\frac{1}{1+l}.\end{aligned}$$

In order to prove the upper bound, we use the left-hand side of Lemma 3.1.2:

$$\begin{aligned}\sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)} &\leq \frac{4}{\mu_{0l}^2 - l^2 - l(d-2)} + \sum_{k\geq 4} \frac{1}{(l+\frac{d-2}{2}+\pi(k-2))^2 - l^2 - l(d-2)} \\ &\leq \frac{4}{\mu_{0l}^2 - l^2 - l(d-2)} + \sum_{k\geq 2}\frac{1}{\pi^2k^2 + 2\pi kl} \\ &\leq \frac{4}{\mu_{0l}^2 - l^2 - l(d-2)} + \int_1^\infty dx\,\frac{1}{\pi x\,(\pi x + 2l)}. \end{aligned} \tag{3.9}$$

Let us consider the first term of this expression; denoting by $\{\mu'_{k\nu},\ k\in\mathbb{N}\}$ the list of zeros of J'_ν and using (3.3), we obtain for $d>2$:

$$\mu_{0l} \underset{l\to\infty}{\sim} \mu'_{0,l+\frac{d-2}{2}},$$

and $\mu_{0l} = \mu'_{0l}$ by definition when $d=2$. By [1, 9.5.16]), there exists now $c \cong 0.80861 > 0$ such that

$$\mu'_{0\nu} \underset{\nu\to\infty}{\sim} \nu + c\,\nu^{\frac{1}{3}},$$

therefore,

$$\frac{1}{\mu_{0l}^2 - l^2 - l(d-2)} \underset{l\to\infty}{\sim} \frac{1}{(l+\frac{d-2}{2}+c\,(l+\frac{d-2}{2})^{\frac{1}{3}})^2 - l^2 - l(d-2)} \underset{l\to\infty}{\sim} \frac{C}{l^{\frac{4}{3}}}.$$

Computing now the second term in (3.9) by simple element decomposition gives

$$\int_1^\infty dx\,\frac{1}{\pi x\,(\pi x+2l)} = \frac{1}{2\pi l}\,\ln\left(\frac{x}{x+\frac{2l}{\pi}}\right)\Bigg|_1^\infty = \frac{\ln(1+\frac{2l}{\pi})}{2\pi l} \leq C_2\,\frac{\ln(1+l)}{1+l},$$

which dominates $\frac{C}{l^{\frac{4}{3}}}$ for sufficiently large l, so the conclusion follows. □

We can now state the following theorem, which is a reformulation of Theorems 2.5.3 and 2.5.4 in the present setting.

Theorem 3.3.3. *Let* $(u_0, v_0) \in L^2(B(0,1)) \oplus H^{-1}(B(0,1))$. *If*

$$\sum_{l \in \mathbb{N}} \frac{a_l \, \ln(1+l)}{1+l} < \infty, \tag{3.10}$$

then there exists a unique weak solution u *of equation (3.8) such that* $\mathbb{E}(\|u(t)\|_0^2) < \infty$, *for all* $t \in \mathbb{R}_+$. *On the other hand, if there exists a weak solution* u *to equation (3.8) such that* $\mathbb{E}(\|u(t_0)\|_0^2) < \infty$, *for some* $t_0 > 0$, *then*

$$\sum_{l \in \mathbb{N}} \frac{a_l}{1+l} < \infty. \tag{3.11}$$

Proof. Let us first prove the sufficiency of (3.10). By Theorem 2.5.3, we simply have to check that this condition implies parts (i) and (ii) of Assumption H_0 of the preceding chapter. By the comment made at the end of Section 3.1, we see that Γ_D is continuous with respect to the H^1-norm on $B(0,1)$ if and only if Γ_S is continuous with respect to the $H^{\frac{1}{2}}$-norm on S^{d-1}. Let us check the latter. (3.10) certainly implies that there exists $C > 0$ such that

$$a_l \leq C \; (1+l), \qquad \forall l \in \mathbb{N}.$$

Let then $\varphi \in C^\infty(S^{d-1})$; as already mentioned in this chapter, φ can be written as

$$\varphi = \sum_{l \in \mathbb{N}} \sum_{m=1}^{N(d,l)} c_{lm} \; Y_l^m, \quad \text{where} \quad \|\varphi\|_{\frac{1}{2}}^2 = \sum_{l \in \mathbb{N}} (1+l) \sum_{m=1}^{N(d,l)} |c_{lm}|^2 < \infty.$$

By (3.7), we have

$$\begin{aligned} \Gamma_S(\varphi, \varphi) &= \sum_{l \in \mathbb{N}} a_l \, \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \\ &\leq C \; |S^{d-1}| \sum_{l \in \mathbb{N}} (1+l) \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \\ &= C \; |S^{d-1}| \; \|\varphi\|_{\frac{1}{2}}^2, \end{aligned}$$

where we have used the above estimate on a_l and the fact that $N(d,l) \geq 1$ for all $l \in \mathbb{N}$. So part (i) of assumption H_0 is satisfied. Part (ii) of this assumption is then a direct consequence of condition (3.10), Lemmas 3.3.1 and the upper bound in Lemma 3.3.2.

On the other hand, in order to show the necessity of condition (3.11), we use Theorem 2.5.4, which states the necessity of part (ii) of Assumption H_0. By Lemma 3.3.1 and the lower bound in Lemma 3.3.2, we obtain directly the necessity of condition (3.11), so the theorem is proven. ☐

Remark 3.3.4. The difference between conditions (3.10) and (3.11) comes from the estimate on the zeros of Bessel functions of Lemma 3.1.2. Since this estimate seems to be the best available among uniform estimates in k and l, it seems difficult to fill in the gap and decide which of the two conditions (3.10) and (3.11) is optimal. Still, one can say something more about condition (3.11). By the proof of the preceding theorem, we see that this condition implies that Γ_S is continuous with respect to the $H^{\frac{1}{2}}$-norm, and this in turn implies that there exists $Q_S \in \mathcal{L}(H^{\frac{1}{2}}(S^{d-1}))$, the space of linear continuous operators on $H^{\frac{1}{2}}(S^{d-1})$, such that

$$\Gamma_S(\varphi,\psi) = \langle \varphi, Q_S \psi \rangle_{\frac{1}{2}}, \qquad \forall \varphi,\psi \in C^\infty(S^{d-1}).$$

But condition (3.11) then simply says that $Q_S \in \mathcal{L}^1(H^{\frac{1}{2}}(S^{d-1}))$, the space of trace-class linear operators on $H^{\frac{1}{2}}(S^{d-1})$, that is,

$$\sum_{n\in\mathbb{N}} \langle g_n, Q_S g_n \rangle_{\frac{1}{2}} < \infty,$$

where $\{g_n,\ n \in \mathbb{N}\}$ is any Hilbertian basis of $H^{\frac{1}{2}}(S^{d-1})$.

Condition (3.11) can therefore be expressed in a general way which could be adapted to a noise concentrated on a boundary with a different shape. It then seems to be more natural than (3.10) (and so the lower bound in Lemma 3.1.2 is perhaps not optimal). In Appendix C, we will see that in the case of a noise concentrated on one side of a hypercube, we obtain a necessary and sufficient condition which can be expressed in the same general way as condition (3.11).

Remark 3.3.5. Following Remark 2.5.6 of the preceding chapter, we see that if the solution u of equation (2.4) would take its values in $H^1(B(0,1))$, then condition (2.20) would be satisfied, which can be rewritten in the present case as

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \gamma_{klm} < \infty.$$

By the preceding calculations, we have

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \gamma_{klm} = |S^{d-1}| \sum_{l\in\mathbb{N}} a_l \left(\sum_{k\in\mathbb{N}} f_{kl}(1)^2 \right),$$

and since the term in parentheses is never finite, this allows us to conclude that this condition is never satisfied, so there also never exists a solution with values in $H^1(B(0,1))$, when the noise under consideration is a boundary noise.

Remark 3.3.6. Now that we have obtained an explicit condition which guarantees the existence of a solution u with values in $L^2(B(0,1))$, we could consider non-linear equations of the same type, following the general theory of G. Da Prato and J. Zabczyk. For the heat equation, this has been already studied in [34]. However, note that it is impossible to consider non-linear terms of the form $g(u(t,x))\ \dot{F}^S(t,x)$, since the noise $\dot{F}^S$ is concentrated on the boundary but the solution u is not well defined on that boundary.

3.4 Reformulation of the necessary condition in the case $d = 2$

We consider here the case $d = 2$, that is, the linear hyperbolic equation with two space dimensions driven by noise concentrated on the unit circle S^1. Since S^1 is a group, this case is special and we can therefore use Fourier analysis techniques to reformulate condition (3.11). In order to do this, one needs a further assumption on the covariance Γ_S. Let us first recall that this covariance is given by

$$\begin{aligned}\Gamma_S(\varphi,\psi) &= \sum_{l\in\mathbb{N}} a_l \int_{S^1} d\sigma(x) \int_{S^1} d\sigma(y)\ \varphi(x)\ P_l(x\cdot y)\ \psi(y) \\ &= \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta_x \int_{-\pi}^{\pi} d\theta_y\ \varphi(\theta_x)\cos(l(\theta_x-\theta_y))\ \psi(\theta_y) \\ &= \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\ \cos(l\theta) \int_{-\pi}^{\pi} d\theta_x\ \varphi(\theta_x)\ \psi(\theta_x-\theta),\end{aligned}$$

by the change of variable $\theta = \theta_x - \theta_y$. This can be rewritten as

$$\Gamma_S(\varphi,\psi) = \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\ \cos(l\theta)\ (\varphi * \tilde{\psi})(\theta),$$

where

$$(\varphi * \psi)(\theta) = \int_{-\pi}^{\pi} d\theta' \varphi(\theta')\ \ \psi(\theta-\theta'),$$

is the convolution product on S^1 and $\tilde{\psi}(\theta) = \psi(-\theta)$. The map

$$\varphi \mapsto \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\ \cos(l\theta)\ \varphi(\theta), \qquad \varphi \in C^\infty(S^1),$$

defines a distribution on S^1 (see [57, Chap. VII, §I]). Let us now assume that this distribution is non-negative. By the fundamental theorem of Radon-Riesz (see for example [32, Chap. II, Thm 2.2]), there exists therefore a non-negative Borel measure Γ on S^1 such that

$$\int_{-\pi}^{\pi} \Gamma(d\theta)\ \varphi(\theta) = \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\ \cos(l\theta)\ \varphi(\theta), \qquad \forall \varphi \in C^\infty(S^1).$$

We now have the following reformulation of condition (3.11) as a condition on the measure Γ.

Proposition 3.4.1. *If condition (3.11) is satisfied, then*

$$\int_{-\pi}^{\pi} \Gamma(d\theta)\ \ln\left(\frac{1}{|\theta|}\right) < \infty. \tag{3.12}$$

This condition is actually a condition on the integrability of Γ near zero, as are the conditions obtained for Gaussian noises on $\mathbb{R}^d$ in the following chapters. Note that since we have proven that (3.11) is a necessary condition, but not that it is a sufficient one, there would be little interest in considering the converse implication.

Moreover, note that if $\Gamma(d\theta) = f(|\theta|)\ d\theta$, with f a continuous function on $]0,\pi]$, condition (3.12) simply reads

$$\int_0^\pi d\theta\ f(\theta)\ \ln\left(\frac{1}{\theta}\right) < \infty.$$

Proof. Set

$$h(\theta) = \ln\left(\frac{1}{|\theta|}\right) = \frac{c_0}{2} + \sum_{l\in\mathbb{N}^*} c_l\ \cos(l\theta), \qquad \theta \in [-\pi,\pi]\backslash\{0\}.$$

Note that since h belongs to $L^2(-\pi,\pi)$, the above Fourier series converges also in $L^2(-\pi,\pi)$. Moreover, computing the coefficient c_l gives, for $l \in \mathbb{N}^*$,

$$\begin{aligned} c_l &= \frac{2}{\pi}\int_0^\pi d\theta\ \ln\left(\frac{1}{\theta}\right)\ \cos(l\theta) \\ &= \frac{2}{\pi}\ \ln\left(\frac{1}{\theta}\right)\ \frac{\sin(l\theta)}{l}\Big|_0^\pi + \frac{2}{\pi}\int_0^\pi d\theta\ \frac{\sin(l\theta)}{l\theta} \\ &= \frac{2}{\pi l}\int_0^{l\pi} du\ \frac{\sin(u)}{u}, \end{aligned}$$

by integration by parts and change of variable $u = l\theta$. Since

$$\int_0^{l\pi} du\ \frac{\sin(u)}{u} \underset{l\to\infty}{\to} \frac{\pi}{2},$$

we obtain that $c_l \sim \frac{1}{l}$ as $l \to \infty$. We therefore have

$$\sum_{l\in\mathbb{N}} \frac{a_l}{1+l} < \infty \quad \text{if and only if} \quad \sum_{l\in\mathbb{N}} a_l\ c_l < \infty.$$

But this last condition implies, by the dominated convergence theorem, that

$$\lim_{t\downarrow 0} \sum_{l\in\mathbb{N}} a_l\ c_l\ e^{-lt} < \infty. \tag{3.13}$$

Let us now define, for $t > 0$,

$$\begin{aligned} \psi_t(\theta) &= \frac{1}{\pi} + \frac{2}{\pi}\sum_{l\in\mathbb{N}^*} e^{-lt}\ \cos(l\theta) \\ &= \frac{1}{\pi}\ \frac{\sinh(t)}{\cosh(t) - \cos(\theta)}, \qquad \theta \in [-\pi,\pi], \end{aligned}$$

by a direct calculation. It is a non-negative function on $[-\pi,\pi]$ and since $e^{-lt} \to 1$, $\lim_{t\downarrow 0}\psi_t = \delta_0$. Let us now define

$$\varphi_t(\theta) = (h * \psi_t)(\theta), \qquad \theta \in [-\pi,\pi].$$

By Parseval's identity,

$$\varphi_t(\theta) = c_0 + 2\sum_{l\in\mathbb{N}^*} c_l\ e^{-lt}\ \cos(l\theta), \qquad \theta \in [-\pi,\pi],$$

and belongs to $C^\infty(S^1)$, for all $t > 0$, since the coefficients $c_l\, e^{-lt}$ are rapidly decreasing in l. Moreover,

$$\int_0^\pi \Gamma(d\theta)\ \varphi_t(\theta) = \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\ \cos(l\theta)\ \varphi_t(\theta) = \pi\ \left(a_0\ c_0 + \sum_{l\in\mathbb{N}^*} a_l\ c_l\ e^{-lt}\right). \tag{3.14}$$

Using now (3.13), (3.14) and Fatou's lemma, we obtain that

$$\infty > \lim_{t\downarrow 0} \int_{-\pi}^{\pi} \Gamma(d\theta)\ \varphi_t(\theta) \geq \int_{-\pi}^{\pi} \Gamma(d\theta)\ \liminf_{t\downarrow 0} \varphi_t(\theta),$$

and since

$$\lim_{t\downarrow 0} \varphi_t(\theta) = \ln\left(\frac{1}{\theta}\right), \qquad \forall\theta \in [-\pi,\pi]\backslash\{0\},$$

this proves the result. □

3.5 Noise on a sphere of lower radius

Let us turn back to the beginning of this chapter, but assume now that the noise is concentrated on a sphere of lower radius $r_0 \in]0,1[$, therefore interior to the domain $B(0,1)$. The preceding analysis can also be applied to this case and the changes are the following. The general form for the covariance of the noise is

$$\Gamma_S(\varphi,\psi) = \sum_{l\in\mathbb{N}} a_l\ \Gamma_l(\varphi,\psi),$$

where

$$\Gamma_l(\varphi,\psi) = \int_{S(r_0)} d\sigma(x) \int_{S(r_0)} d\sigma(y)\ \varphi(x)\ P_l\left(d, \frac{x\cdot y}{r_0^2}\right)\ \psi(y),$$

and the a_l and P_l are the same as before and $S(r_0)$ is the sphere of radius r_0. The covariance Γ_D is related to Γ_S by the following:

$$\Gamma_D(\varphi,\psi) = \Gamma_S\left(\varphi\big|_{S(r_0)}, \psi\big|_{S(r_0)}\right), \qquad \varphi,\psi \in \mathcal{S}(D).$$

Performing the same calculations as before, we obtain that

$$\gamma_{klm} = \Gamma_D(e_{klm}, e_{klm}) = f_{kl}(r_0)^2\ a_l\ \frac{|S^{d-1}|}{N(d,l)},$$

where (see Lemma 3.1.1)

$$f_{kl}(r_0)^2 = \frac{2\ \lambda_{kl}}{\lambda_{kl} - l^2 - l(d-2)}\ \left(\frac{J_l(d,\mu_{kl}\, r_0)}{J_l(d,\mu_{kl})}\right)^2.$$

Therefore, the following condition is satisfied:

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \frac{\gamma_{klm}}{1+\lambda_{kl}} < \infty,$$

if and only if

$$\sum_{l\in\mathbb{N}} a_l \left(\sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)} \left(\frac{J_l(d, \mu_{kl} r_0)}{J_l(d, \mu_{kl})} \right)^2 \right) < \infty.$$

Since

$$J_l(d, r) \underset{r\to\infty}{\sim} \frac{\cos(r - \frac{l\pi}{2} - \frac{\pi}{4})}{\sqrt{r}},$$

(see formula 9.2.1 in [1]), we obtain that the term

$$\left(\frac{J_l(d, \mu_{kl} r_0)}{J_l(d, \mu_{kl})} \right)^2$$

oscillates between 0 and $\frac{1}{r_0}$ as $k \to \infty$. It is therefore difficult to decide whether or not it changes the behavior in l of the sum

$$\sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)} \left(\frac{J_l(d, \mu_{kl} r_0)}{J_l(d, \mu_{kl})} \right)^2,$$

compared to the case where $r_0 = 1$ studied before. Our guess is that the behavior in l does not change, and therefore that the conclusion remains the same, but no explicit calculation has been made in order to check this point.

Chapter 4

Preliminaries for the study in $\mathbb{R}^d$

4.1 Tempered distributions and Fourier transform

Fix d a positive natural number and let us introduce the following notations.

- $\mathcal{B}_b(\mathbb{R}^d)$ denotes the set of bounded Borel subsets of $\mathbb{R}^d$.

- For $r > 0$ and $a \in \mathbb{R}^d$, $B(a, r)$ denotes the ball of center a and radius r in $\mathbb{R}^d$.

- $C_0^\infty(\mathbb{R}^d)$ denotes the space of complex-valued C^∞ functions on $\mathbb{R}^d$ with compact support.

- $\mathcal{S}(\mathbb{R}^d)$ denotes the space of complex-valued C^∞ functions on $\mathbb{R}^d$ with rapid decrease.

- $O_M(\mathbb{R}^d)$ denotes the space of complex-valued C^∞ functions on $\mathbb{R}^d$ with polynomial growth.

- $\mathcal{E}'(\mathbb{R}^d)$ denotes the space of distributions with compact support on $\mathbb{R}^d$.

- $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions on $\mathbb{R}^d$ (which is the dual of $\mathcal{S}(\mathbb{R}^d)$).

- $O'_C(\mathbb{R}^d)$ denotes the space of distributions with rapid decrease on $\mathbb{R}^d$ (which is *not* the dual of $O_M(\mathbb{R}^d)$).

- $\mathcal{F}\varphi$ denotes the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^d)$, which is defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} dx\ \varphi(x)\ e^{i\xi\cdot x}, \qquad \xi \in \mathbb{R}^d,$$

and we have the following Fourier inversion formula (cf. [57, formula (VII,2;3)]):

$$\mathcal{F}^{-1}\varphi(\xi) = \frac{1}{(2\pi)^d}\ \mathcal{F}\varphi(-\xi), \qquad \forall \xi \in \mathbb{R}^d. \tag{4.1}$$

- $\mathcal{F}T$ denotes the Fourier transform of $T \in \mathcal{S}'(\mathbb{R}^d)$, which is defined by $\langle \mathcal{F}T, \varphi\rangle = \langle T, \mathcal{F}\varphi\rangle$ for

$\varphi \in \mathcal{S}(\mathbb{R}^d)$. By [57, Chap. VII, Thm XV], we have

$$T \in O'_C(\mathbb{R}^d) \text{ if and only if } \mathcal{F}T \in O_M(\mathbb{R}^d). \tag{4.2}$$

- $\varphi * \psi$ denotes the convolution product of $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, which is defined by

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^d} dy\ \varphi(y)\ \psi(x-y), \qquad x \in \mathbb{R}^d.$$

- $S * T$ denotes the convolution product of $S \in O'_C(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$; it belongs in general to $\mathcal{S}'(\mathbb{R}^d)$, and it belongs to $\mathcal{S}(\mathbb{R}^d)$ if $T \in \mathcal{S}(\mathbb{R}^d)$; moreover, by [57, Chap. VII, Thm XV], we have the following property:

$$\mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T. \tag{4.3}$$

- For $\xi \in \mathbb{R}^d$, δ_ξ denotes the Dirac measure at point ξ and χ_ξ the function defined by $\chi_\xi(x) = e^{i\xi\cdot x}$, $x \in \mathbb{R}^d$ (note that $\mathcal{F}\delta_\xi = \chi_\xi$, so $\mathcal{F}\chi_\xi = (2\pi)^d\ \delta_{-\xi}$ by (4.1)).

- Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be such that ψ is non-negative, supp $\psi \subset \overline{B(0,1)}$, $\int_{\mathbb{R}^d} dx\ \psi(x) = 1$ and define $\psi_n(x) = n^d\ \psi(nx)$, $x \in \mathbb{R}^d$; then (ψ_n) is a sequence of (non-negative and compactly supported) approximations of the Dirac measure δ_0 in the sense that $\psi_n \underset{n\to\infty}{\to} \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$. Moreover, for all $\xi \in \mathbb{R}^d$, $\mathcal{F}\psi_n(\xi) \underset{n\to\infty}{\to} 1$ and $|\mathcal{F}\psi_n(\xi)| \leq 1$ for all n.

Let us also mention the two following facts.

- If $T \in \mathcal{S}'(\mathbb{R}^d)$ is *non-negative* (in the sense that $T(\varphi) \geq 0$ for all $\varphi \geq 0$), then it is also a non-negative *measure* on $\mathbb{R}^d$ (see [57, Chap. I, Thm V]).

- If μ is a signed Borel measure on $\mathbb{R}^d$ with total variation measure $|\mu|$ which is moreover assumed to be tempered, that is, there exists $r > 0$ with

$$\int_{\mathbb{R}^d} \frac{|\mu|(d\xi)}{(1+|\xi|)^r} < \infty,$$

then $\mu \in \mathcal{S}'(\mathbb{R}^d)$ (see [57, Chap. VII, Thm VII]). By extension, we will call μ itself a tempered measure.

4.2 Sobolev spaces

For $\beta \in \mathbb{R}$, let us denote by $H^\beta(\mathbb{R}^d)$ the fractional Sobolev space of order β on $\mathbb{R}^d$, which is the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ whose Fourier transform $\mathcal{F}u$ belongs to $L^2(\mathbb{R}^d; (1+|\xi|^2)^\beta d\xi)$ (see for example [59, p. 251] for an overview of the properties that follow). We define the following scalar product on $H^\beta(\mathbb{R}^d)$:

$$\langle u, v\rangle_\beta = \int_{\mathbb{R}^d} d\xi\ (1+|\xi|^2)^\beta\ \mathcal{F}u(\xi)\ \overline{\mathcal{F}v(\xi)}, \tag{4.4}$$

and denote its corresponding norm $\|\cdot\|_\beta$. Note that for $\beta = 0$, $H^\beta(\mathbb{R}^d)$ is the usual space $L^2(\mathbb{R}^d)$, which we identify to its dual.

Let $n \in \mathbb{N}$. We have the following inclusions:

$$\mathcal{S}(\mathbb{R}^d) \subset \ldots \subset H^n(\mathbb{R}^d) \subset \ldots \subset L^2(\mathbb{R}^d) \subset \ldots \subset H^{-n}(\mathbb{R}^d) \subset \ldots \subset \mathcal{S}'(\mathbb{R}^d)$$

and

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{n\in\mathbb{N}} H^n(\mathbb{R}^d), \qquad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{n\in\mathbb{N}} H^{-n}(\mathbb{R}^d).$$

For $m > n + \frac{d}{2}$, there is an Hilbert-Schmidt imbedding of $H^m(\mathbb{R}^d)$ into $H^n(\mathbb{R}^d)$, which means that for any a Hilbertian basis $\{\varphi_k,\ k \geq 1\}$ of $H^m(\mathbb{R}^d)$, we have

$$\sum_{k\geq 1} \|\varphi_k\|_n^2 < \infty. \tag{4.5}$$

Moreover, note that the following norm on $H^{-n}(\mathbb{R}^d)$:

$$|||u|||_{-n} = \sup_{\varphi\in H^n(\mathbb{R}^d),\varphi\neq 0} \frac{|\langle u,\varphi\rangle|}{\|\varphi\|_n}, \tag{4.6}$$

is equivalent to $\|\cdot\|_{-n}$, so $H^{-n}(\mathbb{R}^d)$ is the dual space of $H^n(\mathbb{R}^d)$.

4.3 Gaussian noises on $\mathbb{R}^d$ and their spectral measure

In this section, we present some general considerations on spatially homogeneous Gaussian noises on $\mathbb{R}^d$, which will be used in Chapters 6 to 8. Formally, such a noise is a generalized centered Gaussian process $\dot F = \{\dot F(x),\ x \in \mathbb{R}^d\}$ whose covariance is given by

$$\mathbb{E}(\dot F(x)\ \dot F(y)) = \Gamma(x-y), \qquad x, y \in \mathbb{R}^d,$$

where Γ is a non-negative definite distribution on $\mathbb{R}^d$. Since $\dot F(x)$ is not well defined for fixed $x \in \mathbb{R}^d$, we will rather consider in the following the process $F = \{F(\varphi),\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ which is related to $\dot F$ by the informal relationship

$$F(\varphi) = \int_{\mathbb{R}^d} dx\ \dot F(x)\ \varphi(x), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{4.7}$$

In order for F to be well defined, we need to assume that the covariance Γ belongs to $\mathcal{S}'(\mathbb{R}^d)$ and that it is *non-negative definite on* $\mathbb{R}^d$, that is,

$$\Gamma(\varphi * \tilde\varphi) \geq 0, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{4.8}$$

where $\tilde\varphi(x) = \overline{\varphi(-x)}$, $x \in \mathbb{R}^d$. By the Kolmogorov extension theorem (see [42, Prop. 3.4]), there exist a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a centered Gaussian process $F = \{F(\varphi),\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ defined on this space, whose covariance is given by

$$\mathbb{E}(F(\varphi)\ \overline{F(\psi)}) = \Gamma(\varphi * \tilde\psi), \qquad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d).$$

If one wants to analyze the properties of the covariance Γ, it is often useful to refer to its spectral measure μ, which is defined in the following generalization of the Bochner theorem due to L. Schwartz (see [57, Chap. VII, Thm XVIII]):

Theorem 4.3.1. *Let $\Gamma \in \mathcal{S}'(\mathbb{R}^d)$. Then Γ is non-negative definite on $\mathbb{R}^d$ if and only if there exists a non-negative tempered Borel measure μ on $\mathbb{R}^d$ such that $\Gamma = \mathcal{F}\mu$.*

Note moreover that when the distribution Γ is real-valued (in the sense that $\Gamma(\varphi) \in \mathbb{R}$ for all real-valued $\varphi \in \mathcal{S}(\mathbb{R}^d)$), then μ is symmetric on $\mathbb{R}^d$, and reciprocally.

In the present work, we consider partial differential equations driven by noises whose spatial component is a process of the form mentioned above. Our study will lead to conditions on the spectral measure of the noise which ensure some regularity of the solution of the equation. A typical condition will be

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty, \tag{4.9}$$

where $\eta \in [0,1]$. We want here to make some comments on this condition, in order to interpret it as condition on the covariance Γ.

4.4 Reformulation of the conditions on the spectral measure

Note that condition (4.9) states that μ needs to decrease sufficiently rapidly at infinity (that is, there are not too many high frequencies in the noise), which can be reformulated into a condition on the integrability of the covariance Γ near 0 (which in turn means that the noise has some regularity) in the case where Γ is a non-negative distribution on $\mathbb{R}^d$ (which implies that it is also a non-negative measure on $\mathbb{R}^d$).

The first case that we consider is the case $\eta = 0$. In this case, condition (4.9) says that μ is a finite measure, and therefore, by the classical Bochner theorem, it is equivalent to say that Γ is a uniformly continuous and bounded function on $\mathbb{R}^d$.

The second case is the case $\eta = 1$. As mentioned in the intoduction, this case has been studied in [30] and the conclusion is that when Γ is assumed to be a non-negative distribution on $\mathbb{R}^d$, (4.9) is equivalent to

$$\begin{cases} \text{no condition on } \Gamma, & \text{when d=1,} \\ \displaystyle\int_{B(0,1)} \Gamma(dx)\ \ln\left(\frac{1}{|x|}\right) < \infty, & \text{when } d = 2, \\ \displaystyle\int_{B(0,1)} \Gamma(dx)\ \frac{1}{|x|^{d-2}} < \infty, & \text{when } d \geq 3. \end{cases} \tag{4.10}$$

Let us finally consider the case $\eta \in]0,1[$ and assume that Γ is a non-negative distribution on $\mathbb{R}^d$. Let us then define

$$G_{d,\eta}(x) = \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x), \qquad x \in \mathbb{R}^d.$$

Since $\mathcal{F}G_{d,\eta}$ depends only on $|\xi|$, we have, by [58, (V,3;22)],

$$G_{d,\eta}(x) = \frac{1}{(2\pi)^{\frac{d}{2}}\ |x|^{\frac{d-1}{2}}} \int_0^\infty dr\ r^{\frac{d-1}{2}}\ \frac{1}{(1+r^2)^\eta}\ \sqrt{|x|\,r}\ J_{\frac{d-2}{2}}(|x|\,r),$$

where J_ν is the regular Bessel function of the first kind and of order ν (see Appendix B for a definition). Using [46, formula I.4.23], we obtain that there exists a constant $C_{d,\eta} > 0$ such that

$$G_{d,\eta}(x) = C_{d,\eta}\ |x|^{\eta-\frac{d}{2}}\ K_{\frac{d}{2}-\eta}(|x|), \tag{4.11}$$

where K_ν is the modified Bessel function of the second kind and of order ν (see also Appendix B for a definition). Let us moreover define

$$F_{d,\eta}(y) = \int_{\mathbb{R}^d} \Gamma(dx)\ G_{d,\eta}(x-y), \qquad y \in \mathbb{R}^d.$$

We can now formulate the following proposition.

Proposition 4.4.1. *Let us assume that Γ is a non-negative measure on $\mathbb{R}^d$. If condition (4.9) is satisfied, then*

$$F_{d,\eta}(0) = \int_{\mathbb{R}^d} \Gamma(dx)\ G_{d,\eta}(x) < \infty. \tag{4.12}$$

On the other hand, if $F_{d,\eta}$ is a bounded function on $\mathbb{R}^d$ (which implies that $F_{d,\eta}(0) < \infty$), then condition (4.9) is satisfied.

Remark 4.4.2. Note that $F_{d,\eta}$ is non-negative definite, since its Fourier transform

$$\mathcal{F}F_{d,\eta} = \mathcal{F}\left(\Gamma * G_{d,\eta}\right) = \mu \cdot \mathcal{F}G_{d,\eta},$$

is non-negative. Therefore, the assumption that $F_{d,\eta}$ is a bounded function is not a particularly strong assumption, since every non-negative definite distribution which is continuous at 0 is a bounded function by [57, Chap. VII, p. 276].

Proof of Proposition 4.4.1. Suppose first that condition (4.9) is satisfied. We then have by the dominated convergence theorem,

$$\infty > \int_{\mathbb{R}^d} \mu(d\xi)\ \frac{1}{(1+|\xi|^2)^\eta} = \lim_{t\downarrow 0} \int_{\mathbb{R}^d} \mu(d\xi)\ \frac{1}{(1+|\xi|^2)^\eta}\ e^{-t\,|\xi|^2}.$$

Let us denote $p_t = \mathcal{F}^{-1}(e^{-t\,|\xi|^2})$, the heat kernel in $\mathbb{R}^d$. By standard properties of the Fourier transform, we have

$$\int_{\mathbb{R}^d} \mu(d\xi)\ \frac{1}{(1+|\xi|^2)^\eta}\ e^{-t\,|\xi|^2} = \int_{\mathbb{R}^d} \Gamma(dx)\ (G_{d,\eta} * p_t)(x), \tag{4.13}$$

and Fatou's lemma tells us that

$$\liminf_{t\downarrow 0} \int_{\mathbb{R}^d} \Gamma(dx)\ (G_{d,\eta} * p_t)(x) \geq \int_{\mathbb{R}^d} \Gamma(dx)\ G_{d,\eta}(x),$$

since

$$(G_{d,\eta} * p_t)(x) \underset{t\downarrow 0}{\to} G_{d,\eta}(x), \qquad \forall x \neq 0.$$

This proves the first statement of the theorem. In order to prove the second one, let us assume that $F_{d,\eta}$ is bounded, and note that since p_t is a probability measure on $\mathbb{R}^d$ for all $t \in \mathbb{R}_+$, we have

$$\sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}^d} dy\ p_t(y)\ F_{d,\eta}(y) \leq \sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) < \infty.$$

But on the other hand,

$$\int_{\mathbb{R}^d} dy\ p_t(y)\ F_{d,\eta}(y) = \int_{\mathbb{R}^d} \Gamma(dx)\ (G_{d,\eta} * p_t)(x),$$

by definition of $F_{d,\eta}$ and Fubini's theorem. By (4.13), this expression is still equal to

$$\int_{\mathbb{R}^d} \mu(d\xi)\ \frac{1}{(1+|\xi|^2)^\eta}\ e^{-t\,|\xi|^2} \underset{t\downarrow 0}{\to} \int_{\mathbb{R}^d} \mu(d\xi)\ \frac{1}{(1+|\xi|^2)^\eta},$$

by the monotone convergence theorem, so the theorem is proven. □

Remark 4.4.3. In [54, Prop. 5.3], (4.9) and (4.12) were announced to be equivalent. But at the end of the proof of Lemma 5.1 in [54], the following equality:

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} \Gamma(dx)\ (G_{d,\eta} * p_t)(x) = \int_{\mathbb{R}^d} \Gamma(dx)\ G_{d,\eta}(x),$$

was claimed to be true because of the monotone convergence theorem and the fact that the map

$$t \mapsto e^{-t}\ (G_{d,\eta} * p_t)(x) = C_\eta \int_t^\infty d\sigma\ e^{-\sigma}\ (\sigma - t)^{\eta-1}\ p_\sigma(x)$$

is monotone. If this map is indeed increasing as $t \downarrow 0$ when $\eta = 1$ (which yields the characterization (4.10)), this is no longer the case for $\eta \in\ ]0,1[$, so the proof of Proposition 5.3 in [54] seems to be incomplete.

Let us now make (4.12) more explicit. Since Γ is a tempered measure on $\mathbb{R}^d$ and using estimate (B.1), we see that the integral over $B(0,1)^c$ in (4.12) is always finite, so we can omit this part of the integral. On the other hand, using estimate (B.2), we obtain the following equivalences for condition (4.12).

- If $\eta > \frac{d}{2}$, then (4.12) imposes no particular restriction on Γ.

- If $\eta = \frac{d}{2}$, then (4.12) is equivalent to

$$\int_{B(0,1)} \Gamma(dx)\ \ln\left(\frac{1}{|x|}\right) < \infty.$$

- If $\eta < \frac{d}{2}$, then (4.12) is equivalent to

$$\int_{B(0,1)} \Gamma(dx)\ \frac{1}{|x|^{d-2\eta}} < \infty.$$

Remark 4.4.4. Note when $\eta > \frac{d}{2}$, $G_{d,\eta}$ is continuous at 0 and rapidly decreasing on $\mathbb{R}^d$, so it is dominated by a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and we therefore have

$$F_{d,\eta}(y) \le \int_{\mathbb{R}^d} \Gamma(dx)\ \varphi(x-y) = \int_{\mathbb{R}^d} \mu(d\xi)\ \mathcal{F}\varphi(\xi)\ \chi_y(\xi) \le \int_{\mathbb{R}^d} \mu(d\xi)\ \mathcal{F}\varphi(\xi), \quad \forall y \in \mathbb{R}^d,$$

which in turn implies that $F_{d,\eta}$ is bounded and therefore that (4.9) holds for all μ by Proposition 4.4.1.

Remark 4.4.5. When μ is the Lebesgue measure on $\mathbb{R}^d$ (which is the spectral measure of *white noise* on $\mathbb{R}^d$, that is, the noise with covariance $\Gamma = \delta_0$), the integral in condition (4.9) is equal to

$$\int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|^2)^\eta} = C \int_0^\infty dr\ \frac{r^{d-1}}{(1+r^2)^\eta},$$

which is finite if and only if $\eta > \frac{d}{2}$. From this and Remark 4.4.4, we see that if condition (4.9) is satisfied for white noise, then it is satisfied for any noise with non-negative covariance. In this sense, the white noise represents the most irregular noise among noises with non-negative covariance.

Note that when $\Gamma(dx) = f(|x|)\ dx$, with f a continuous function on $]0,\infty[$, we have the following.

- If $\eta > \frac{d}{2}$, then (4.12) imposes no particular restriction on f.

- If $\eta = \frac{d}{2}$, then (4.12) is equivalent to

$$\int_0^1 dr\ f(r)\ r^{d-1}\ \ln\left(\frac{1}{r}\right) < \infty.$$

- If $\eta < \frac{d}{2}$, then (4.12) is equivalent to

$$\int_0^1 dr\ f(r)\ \frac{1}{r^{1-2\eta}} < \infty.$$

In order to be complete, let us finally give a stronger but more explicit sufficient condition on Γ which implies (4.9).

Proposition 4.4.6. *Let us assume that Γ is a non-negative measure on $\mathbb{R}^d$ and let $\eta \le \frac{d}{2}$. If there exists $\gamma > d - 2\eta$ and $C > 0$ such that*

$$\Gamma(B(a,r)) \le C\ r^\gamma, \qquad \forall a \in \mathbb{R}^d,\ r > 0, \tag{4.14}$$

then $F_{d,\eta}$ is bounded and condition (4.9) is therefore satisfied by Proposition 4.4.1.

Proof. Let us decompose $F_{d,\eta}$ in two parts:

$$F_{d,\eta}(y) = \int_{B(y,1)} \Gamma(dx)\ G_{d,\eta}(x-y) + \int_{B(y,1)^c} \Gamma(dx)\ G_{d,\eta}(x-y), \qquad y \in \mathbb{R}^d.$$

By (4.11) and (B.1), we have

$$\begin{aligned}
\int_{B(y,1)^c} \Gamma(dx)\ G_{d,\eta}(x-y) &\le \sum_{n\ge 1} \int_{2^{n-1}\le |x-y| \le \varepsilon\, 2^n} \Gamma(dx)\ C\ e^{-|x-y|} \\
&\le C \sum_{n\ge 1} \Gamma(B(y,2^n))\ \exp(-2^{n-1}) \\
&\le C \sum_{n\ge 1} 2^{n\gamma}\ \exp(-2^{n-1}) < \infty,
\end{aligned}$$

and the bound does not depend on y. Let now $\eta < \frac{d}{2}$. Using (B.2), we obtain

$$\begin{aligned}
\int_{B(y,1)} \Gamma(dx)\ G_{d,\eta}(x-y) &\le \sum_{n\ge 1} \int_{2^{-n}\le |x-y| \le 2^{-n+1}} \Gamma(dx)\ \frac{C}{|x-y|^{d-2\eta}} \\
&\le C \sum_{n\ge 1} \Gamma(B(y,2^{-n+1}))\ 2^{n(d-2\eta)} \\
&\le C \sum_{n\ge 1} 2^{(-n+1)\gamma}\ 2^{n(d-2\eta)} < \infty,
\end{aligned}$$

by the assumption made, and the bound again does not depend on y. This estimate and the previous one prove that $F_{d,\eta}$ is bounded when $\eta < \frac{d}{2}$. When $\eta = \frac{d}{2}$, we have, using again (B.2),

$$\begin{aligned}
\int_{B(y,1)} \Gamma(dx)\ G_{d,\eta}(x-y) &\le \sum_{n\ge 1} \int_{2^{-n}\le |x-y| \le 2^{-n+1}} \Gamma(dx)\ C\ \ln\left(\frac{1}{|x-y|}\right) \\
&\le C \sum_{n\ge 1} \Gamma(B(y,2^{-n+1}))\ \ln(2^n) \\
&\le C \sum_{n\ge 1} 2^{(-n+1)\gamma}\ \ln(2^n) < \infty,
\end{aligned}$$

by the assumption made, so $F_{d,\eta}$ is bounded also in this case, and this completes the proof. □

Let us now consider a class of covariances for which condition (4.9) gives an optimal criterion. Consider that Γ is of the form

$$\Gamma(dx) = \frac{dx}{|x|^\alpha},$$

where $0 < \alpha < d$ (in order for Γ to be a well defined covariance). Let us first make explicit a sufficient condition which implies condition (4.9) by means of Proposition 4.4.6. For all $r > 0$, we have

$$\Gamma(B(0,r)) = C \int_0^r du\ u^{d-1-\alpha} = \frac{C}{d-\alpha}\ r^{d-\alpha}.$$

For $a \in \mathbb{R}^d$ such that $|a| \le 2r$, we have by the triangle inequality,

$$\Gamma(B(a,r)) \le \Gamma(B(0,3r)) = \frac{C\ 3^{d-\alpha}}{d-\alpha}\ r^{d-\alpha}.$$

Finally, for $a \in \mathbb{R}^d$ such that $|a| > 2r$, we have

$$\Gamma(B(a,r)) = \int_{B(a,r)} \frac{dx}{|x|^\alpha} \le \frac{|B(a,r)|}{(|a|-r)^\alpha} \le \frac{\tilde{C}\, r^d}{r^\alpha} = \tilde{C}\, r^{d-\alpha}.$$

Therefore, if $\alpha < 2\eta$ (which is a restriction only when $\eta < \frac{d}{2}$), then (4.14) is satisfied, hence (4.9) by Proposition 4.4.6. On the other hand, if $\alpha \ge 2\eta$, then by estimate (B.2), we have

$$\int_{\mathbb{R}^d} \Gamma(dx)\, G_{d,\eta}(x) \ge C \int_{B(0,1)} \frac{dx}{|x|^\alpha\, |x|^{d-2\eta}} \ge C \int_0^1 \frac{dr}{r} = \infty,$$

so (4.12) is not satisfied, and neither is (4.9) by Proposition 4.4.1. For this simple class of covariances, we have therefore obtained a condition ($\alpha < 2\eta$) equivalent to condition (4.9).

Chapter 5

Linear hyperbolic equation in $\mathbb{R}^d$

Let $a, b \in \mathbb{R}$. We are interested in solving the following stochastic linear hyperbolic equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) + b\,u(t,x) - \Delta u(t,x) = \dot{F}^0(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in \mathbb{R}^d, & \end{cases} \tag{5.1}$$

where u_0, v_0 are two given distributions on $\mathbb{R}^d$ and $\dot{F}^0 = \{\dot{F}^0(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a generalized centered Gaussian process whose covariance is formally given by

$$\mathbb{E}(\dot{F}^0(t,x)\ \dot{F}^0(s,y)) = \delta_0(t-s)\ \Gamma_0(x,y),$$

where δ_0 is the usual Dirac measure on $\mathbb{R}$ and Γ_0 is a non-negative definite measure on $\mathbb{R}^d \times \mathbb{R}^d$, in a sense that will be precised below.

Note that there are three interesting particular cases of this general equation. When $a = b = 0$, this is simply the wave equation. When $a \geq 0$ and $b = 0$, this equation is the wave equation with attenuation, also called the telegraph equation when $d = 1$. And finally, when $a = 0$, this is the Klein-Gordon equation. What will appear in the following is that the values of a and b have no impact on the hyperbolic nature of the equation (that is, the singularity of the Green kernel and the finite speed of propagation of the equation), therefore on the problems studied in the following. On the other hand, Appendix D gives an example of a higher order equation whose nature changes depending on the coefficients.

For technical reasons (see Remark A.1.1), we will restrict ourselves to the two cases where either $d \leq 3$ and a, b are any real numbers, or d is any positive natural number and $a = b = 0$.

5.1 Gaussian noise

Since $\dot{F}^0(t,x)$ is not well defined for fixed $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we will proceed as in Section 4.3 and rather consider in the following the process $F^0 = \{F^0_t(\varphi),\ t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ which is related to $\dot F^0$ by the informal relationship

$$F_t^0(\varphi) = \int_0^t ds \int_{\mathbb{R}^d} dx\ \dot F^0(s,x)\ \varphi(x), \qquad t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{5.2}$$

In order to define F^0 rigorously, we need some precise assumptions on the covariance Γ_0: we assume that it is a signed Borel measure on $\mathbb{R}^d \times \mathbb{R}^d$ (with total variation measure $|\Gamma_0|$) which is also *non-negative definite on* $\mathbb{R}^d \times \mathbb{R}^d$, that is,

$$\sum_{i,j=1}^m c_i\ \overline{c_j}\ \Gamma_0(A_i \times A_j) \geq 0, \qquad \forall m \geq 1,\ c_1, \dots, c_m \in \mathbb{C},\ A_1, \dots, A_m \in \mathcal{B}_b(\mathbb{R}^d),$$

which implies that $\Gamma_0(\cdot,\cdot)$ is hermitian (see [6, p. 68]), hence symmetric, since it is a real-valued measure. Furthermore, we assume that there exists a non-negative Borel measure ν_0 on $\mathbb{R}^d \times \mathbb{R}^d$, which is also non-negative definite, which *dominates* $|\Gamma_0|$, that is,

$$|\Gamma_0|(A \times B) \leq \nu_0(A \times B), \qquad \forall A, B \in \mathcal{B}_b(\mathbb{R}^d),$$

and which is moreover *tempered*, that is, there exists $r > 0$ such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\nu_0(dx,dy)}{(1+|x|+|y|)^r} < \infty.$$

Note that in general, $|\Gamma_0|$ is *not* non-negative definite, even if Γ_0 is; that is why we need a non-negative definite dominating measure ν_0. These assumptions are used in the definition of the stochastic integral with respect to the noise F^0 (see Section 5.3). In Chapter 6, we will see examples of covariances which satisfy such assumptions.

By the Kolmogorov extension theorem (see [42, Prop. 3.4]), there exist a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a centered Gaussian process $F^0 = \{F_t^0(\varphi),\ t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ defined on this space, whose covariance is given by

$$\mathbb{E}(F_t^0(\varphi)\ \overline{F_s^0(\psi)}) = (t \wedge s) \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx,dy)\ \varphi(x)\ \overline{\psi(y)}, \qquad \forall t,s \in \mathbb{R}_+,\ \varphi,\psi \in \mathcal{S}(\mathbb{R}^d). \tag{5.3}$$

We study now the joint regularity in time and space of this process. For this, we need the following two lemmas.

Lemma 5.1.1. *For all $t \in \mathbb{R}_+$, $F_t^0(\cdot)$ is a random linear functional on $\mathcal{S}(\mathbb{R}^d)$, that is, for all $\lambda \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$F_t^0(\lambda\,\varphi + \psi) = \lambda\ F_t^0(\varphi) + F_t^0(\psi), \quad \mathbb{P}-a.s. \tag{5.4}$$

and for all $T > 0$, there exists $C_T > 0$ and $n \in \mathbb{N}$ such that

$$\mathbb{E}(|F_t^0(\varphi) - F_s^0(\varphi)|^2) \leq C_T\ \|\varphi\|_n^2\ |t-s|, \qquad \forall s,t \in [0,T],\ \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{5.5}$$

where $\|\cdot\|_n$ is the Sobolev norm defined by (4.4).

Proof. The first statement follows from formula (5.3) and the fact that the two complex-valued square integrable random variables $Z_1 = F_t^0(\lambda\,\varphi + \psi)$ and $Z_2 = \lambda\; F_t^0(\varphi) + F_t^0(\psi)$ are equal $\mathbb{P}-a.s.$ if and only if $\mathbb{E}(|Z_1|^2) = \mathbb{E}(Z_1\overline{Z_2}) = \mathbb{E}(|Z_2|^2)$. For the second, note that since Γ_0 is a signed tempered Borel measure on $\mathbb{R}^d \times \mathbb{R}^d$, it is in $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ and since

$$\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d) = \bigcup_{n\in\mathbb{N}} H^{-n}(\mathbb{R}^d \times \mathbb{R}^d),$$

there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\;\varphi(x)\;\overline{\varphi(y)} \leq C\;\|\varphi\otimes\overline{\varphi}\|_n, \qquad \forall\varphi\in\mathcal{S}(\mathbb{R}^d).$$

Moreover, using definition (4.4), we see that

$$\begin{aligned}
\|\varphi\otimes\overline{\varphi}\|_n^2 &= \int_{\mathbb{R}^d\times\mathbb{R}^d} d\xi\; d\eta\; (1+|\xi|^2+|\eta|^2)^n\;|\mathcal{F}\varphi(\xi)|^2\;|\mathcal{F}\overline{\varphi}(\eta)|^2\\
&\leq \int_{\mathbb{R}^d} d\xi\;(1+|\xi|^2)^n\;|\mathcal{F}\varphi(\xi)|^2 \int_{\mathbb{R}^d} d\eta\;(1+|\eta|^2)^n\;|\mathcal{F}\varphi(-\eta)|^2\\
&= \|\varphi\|_n^4.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\;\varphi(x)\;\overline{\varphi(y)} \leq C\;\|\varphi\|_n^2. \tag{5.6}$$

This implies that

$$\mathbb{E}(|F_t^0(\varphi) - F_s^0(\varphi)|^2) = |t-s| \int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\;\varphi(x)\;\overline{\varphi(y)} \leq C\;\|\varphi\|_n^2\;|t-s|,$$

which completes the proof. □

In particular, the preceding lemma implies that for all $T > 0$, there exists $C_T > 0$ and $n \in \mathbb{N}$ such that

$$\mathbb{E}(|F_t^0(\varphi)|^2) \leq C_T\;\|\varphi\|_n^2, \qquad \forall t\in[0,T],\;\varphi\in\mathcal{S}(\mathbb{R}^d). \tag{5.7}$$

Remark 5.1.2. The constant C_T does actually not depend on T in the present case. However, in the following, we will refer to properties (5.4) and (5.5) for more general processes. That is why we keep the possibility for the constant to depend on T.

The following lemma is an adaptation of [62, Thm 4.1] in the present simple setting.

Lemma 5.1.3. *Let $F^0 = \{F_t^0(\varphi),\; t\in\mathbb{R}_+,\; \varphi\in\mathcal{S}(\mathbb{R}^d)\}$ be a centered Gaussian process satisfying properties (5.4) and (5.5) of Lemma 5.1.1. Then there exists $m\in\mathbb{N}$ and a modification $\tilde{F}^0$ of F^0 such that for all $t\in\mathbb{R}_+$, $\tilde{F}_t^0(\cdot)\in H^{-m}(\mathbb{R}^d)$.*

Proof. Fix $t\in\mathbb{R}_+$, $m > n + \frac{d}{2}$ and let $\{\varphi_k,\; k\geq 1\}$ be a Hilbertian basis of $H^m(\mathbb{R}^d)$. By (5.7) and (4.5), we obtain that

$$\mathbb{E}\left(\sum_{k\geq 1} |F_t^0(\varphi_k)|^2\right) = \sum_{k\geq 1} \mathbb{E}(|F_t^0(\varphi_k)|^2) \leq C_t \sum_{k\geq 1} \|\varphi_k\|_n^2 < \infty,$$

so the set Ω_1 defined by

$$\Omega_1 = \left\{ \omega \in \Omega : \sum_{k \geq 1} |F_t^0(\varphi_k)|^2 < \infty \right\}$$

has probability one. Therefore, let us define, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\tilde{F}_t^0(\varphi) = \begin{cases} \sum_{k \geq 1} F_t^0(\varphi_k) \langle \varphi_k, \varphi \rangle_m, & \text{on } \Omega_1, \\ 0, & \text{on } \Omega_1^c. \end{cases}$$

This is well defined, since by the Cauchy-Schwarz inequality, we have on Ω_1,

$$\left| \sum_{k \geq 1} F_t^0(\varphi_k) \langle \varphi_k, \varphi \rangle_m \right|^2 \leq \sum_{k \geq 1} |F_t^0(\varphi_k)|^2 \sum_{k \geq 1} |\langle \varphi_k, \varphi \rangle_m|^2 = \sum_{k \geq 1} |F_t^0(\varphi_k)|^2 \, \|\varphi\|_m^2 < \infty. \tag{5.8}$$

It remains to show that $\tilde{F}_t^0$ satisfies the required properties. Let us therefore define, for $N \geq 1$,

$$\varphi^{(N)} = \sum_{k=1}^N \varphi_k \, \langle \varphi_k, \varphi \rangle_m.$$

By (5.4), we have

$$F_t^0(\varphi^{(N)}) = \sum_{k=1}^N F_t^0(\varphi_k) \, \langle \varphi_k, \varphi \rangle_m, \quad \mathbb{P} - a.s.,$$

so we obtain that

$$\begin{aligned} \mathbb{E}\left(\left| F_t^0(\varphi) - \sum_{k=1}^N F_t^0(\varphi_k) \, \langle \varphi_k, \varphi \rangle_m \right|^2 \right) &= \mathbb{E}(|F_t^0(\varphi - \varphi^{(N)})|^2) \\ &\leq C_t \, \|\varphi - \varphi^{(N)}\|_n^2, \end{aligned}$$

by (5.7). Since

$$\|\varphi - \varphi^{(N)}\|_n^2 \leq \|\varphi - \varphi^{(N)}\|_m^2 \underset{N \to \infty}{\longrightarrow} 0,$$

we have proven that

$$F_t^0(\varphi) = \sum_{k \geq 1} F_t^0(\varphi_k) \, \langle \varphi_k, \varphi \rangle_m, \quad \mathbb{P} - a.s.$$

This equality and the fact that $\mathbb{P}(\Omega_1) = 1$ imply that $\tilde{F}^0$ is a modification of F^0. Moreover, $\tilde{F}^0$ takes its values in $H^{-m}(\mathbb{R}^d)$ since by (4.6) and (5.8), we have

$$|||\tilde{F}_t^0|||_{-m}^2 = \sup_{\varphi \in H^m(\mathbb{R}^d), \varphi \neq 0} \frac{|\tilde{F}_t^0(\varphi)|^2}{\|\varphi\|_m^2} \leq \sum_{k \geq 1} |F_t^0(\varphi_k)|^2 \, 1_{\Omega_1} < \infty,$$

which completes the proof. □

This allows us to establish the following theorem.

Proposition 5.1.4. *Let $F^0 = \{F_t^0(\varphi),\ t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ be a centered Gausssian process satisfying properties (5.4) and (5.5) of Lemma 5.1.1. Then F^0 admits a modification $\hat{F}^0$ such that $\hat{F}_t^0(\cdot) \in \mathcal{S}'(\mathbb{R}^d)$ for all $t \in \mathbb{R}_+$ and $\mathbb{P} - a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto \hat{F}_t^0(\varphi)$ is continuous from $\mathbb{R}_+$ to $\mathbb{C}$.*

Proof. The first statement is a direct consequence of the preceding lemma, since $H^{-m}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. In order to prove the second, let us consider again $\{\varphi_k,\ k \geq 1\}$, the Hilbertian basis of $H^{-m}(\mathbb{R}^d)$ used in the proof of the preceding lemma. What we will use here is the Kolmogorov test for Gaussian processes with values in a Hilbert space (see [18, Prop. 3.15]). First note that $\tilde{F}^0$, being the modification of a Gaussian process, is itself a Gaussian process. Moreover, it can also be seen as a Gaussian process with values in $H^{-m}(\mathbb{R}^d)$, with trace-class covariance operator $Q(t)$, since the following condition is satisfied:

$$Tr(Q(t)) = \sum_{k\geq 1} \mathbb{E}(|F_t^0(\varphi_k)|^2) \leq C_t \sum_{k\geq 1} \|\varphi_k\|_n^2 < \infty,$$

by (5.7) and (4.5). Moreover, we have by (4.6) and a slight adaptation of (5.8),

$$\begin{aligned} |||\tilde{F}_t^0 - \tilde{F}_s^0|||_{-m}^2 &= \sup_{\varphi \in H^m(\mathbb{R}^d), \varphi\neq 0} \frac{|\tilde{F}_t^0(\varphi) - \tilde{F}_s^0(\varphi)|^2}{\|\varphi\|_m^2} \\ &\leq \sum_{k\geq 1} |F_t^0(\varphi_k) - F_s^0(\varphi_k)|^2 \; 1_{\Omega_1}. \end{aligned}$$

Therefore, for fixed $T > 0$ and $s, t \in [0, T]$, we also have

$$\begin{aligned} \mathbb{E}(|||\tilde{F}_t^0 - \tilde{F}_s^0|||_{-m}^2) &\leq \sum_{k\geq 1} \mathbb{E}(|F_t^0(\varphi_k) - F_s^0(\varphi_k)|^2) \\ &\leq C_T \sum_{k\geq 1} \|\varphi_k\|_n^2 \; |t-s|, \end{aligned}$$

by (5.5). Since the above sum is finite by (4.5), Proposition 3.15 of [18] states that there exists a modification $\hat{F}^0$ of $\tilde{F}^0$ such that for all $\gamma < \frac{1}{2}$, there exists $C_{T,\gamma} > 0$ which satisfies $\mathbb{P} - a.s.$

$$|||\hat{F}_t^0 - \hat{F}_s^0|||_{-m} \leq C_{T,\gamma}\, |t-s|^\gamma, \qquad \forall s, t \in [0,T],$$

so

$$|\hat{F}_t^0(\varphi) - \hat{F}_s^0(\varphi)| \leq C_{T,\gamma}\, \|\varphi\|_m\, |t-s|^\gamma, \qquad \forall s, t \in [0,T],\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

In particular, this implies that $\mathbb{P} - a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto \hat{F}_t^0(\varphi)$ is continuous from $\mathbb{R}_+$ to $\mathbb{C}$, which ends the proof. □

Remark 5.1.5. The preceding proposition has been expressed in a general way which will be used later on. Note however that the process $\hat{F}^0$ has the more specific property that $\mathbb{P} - a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the process $\{\hat{F}_t^0(\varphi),\ t \in \mathbb{R}_+\}$ is a continuous Brownian motion with covariance parameter

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx, dy)\ \varphi(x)\ \overline{\varphi(y)}.$$

In the following, we will consider implicitely the modification $\hat{F}^0$.

5.2 Weak formulation of the equation

Now that we have a precise definition of the Gaussian noise under consideration, we also need to give a rigorous meaning to equation (5.1). We therefore proceed as in Section 2.3. Setting formally $v(t,x) = \frac{\partial u}{\partial t}(t,x)$, we obtain the following two formal equations, after integration in t of equation (5.1):

$$\begin{cases} u(t,x) = u_0(x) + \displaystyle\int_0^t ds\ v(s,x), \\ v(t,x) = v_0(x) + \displaystyle\int_0^t ds\ (-2a\ v(s,x) - b\ u(s,x) + \Delta u(s,x) + \dot{F}^0(s,x)). \end{cases}$$

We now multiply both sides of these two equations by a test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and integrate them in x, with two more integrations by parts in x of the term with the Laplacian. Assuming that $(u_0, v_0) \in \mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d)$, considering that (u,v) takes its values in $\mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d)$ and using the informal relationship (5.2) gives then the following rigorous formulation: a *weak solution* of equation (5.1) is a process $(u,v) = \{(u(t), v(t)),\ t \in \mathbb{R}_+\}$ with values in $\mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d)$ such that $\mathbb{P} - a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto (\langle u(t), \varphi\rangle, \langle v(t), \varphi\rangle)$ is continuous on $\mathbb{R}_+$ and satisfies, for all $t \in \mathbb{R}_+$,

$$\begin{cases} \langle u(t), \varphi\rangle = \langle u_0, \varphi\rangle + \displaystyle\int_0^t ds\ \langle v(s), \varphi\rangle, \\ \langle v(t), \varphi\rangle = \langle v_0, \varphi\rangle + \displaystyle\int_0^t ds\ (-2a\ \langle v(s), \varphi\rangle - b\ \langle u(s), \varphi\rangle + \langle u(s), \Delta\varphi\rangle) + F_t^0(\varphi). \end{cases} \tag{5.9}$$

Moreover, we say that the weak solution of equation (5.1) is *unique* if for any two solutions $(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$,

$$\langle u^{(1)}(t), \varphi\rangle = \langle u^{(2)}(t), \varphi\rangle \quad \text{and} \quad \langle v^{(1)}(t), \varphi\rangle = \langle v^{(2)}(t), \varphi\rangle,$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathbb{P} - a.s.$

Remark 5.2.1. As for the solution of equation (2.4), we will often be loosely speaking of u, instead of (u,v), for the solution of equation (5.9).

5.3 Stochastic integral

In order to obtain an explicit expression for the solution of (5.9), we shall define a stochastic integral with respect to the noise F^0. This section refers directly to [62, Chap. 2], so some details will be omitted. Consider the augmented natural filtration of the noise, defined by

$$\mathcal{G}_t^0 = \sigma\{F_s^0(\varphi),\ s \in [0,t],\ \varphi \in \mathcal{S}(\mathbb{R}^d)\} \vee \mathcal{N}, \qquad t \in \mathbb{R}_+,$$

where $\mathcal{N}$ is the class of $\mathbb{P}$-null sets in Ω. The noise F^0 extends to a *worthy martingale measure* $M^0 = \{M_t^0(A),\ \mathcal{G}_t^0,\ t \in \mathbb{R}_+,\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$ (see [62, Chap. 2] for a precise definition; in short, M^0

is a martingale in t and a random measure in A) with covariation measure Q_0 and dominating measure K_0 given respectively by

$$Q_0([0,t] \times A \times B) = t\, \Gamma_0(A \times B) \quad \text{and} \quad K_0([0,t] \times A \times B) = t\, \nu_0(A \times B),$$

for $t \in \mathbb{R}_+$ and $A, B \in \mathcal{B}_b(\mathbb{R}^d)$. Note that the existence of the dominating measure K ("dominating" in the sense that $|Q_0([0,t] \times A \times B)| \le K_0([0,t] \times A \times B))$ is a necessary condition for the stochastic integral to be well defined (and also the reason why we say that M^0 is a "worthy" martingale measure).

We can now define the space $\mathcal{E}^0$ of elementary integrands by

$$\mathcal{E}^0 = \Big\{ \phi : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{C} \;\Big|\; \phi(t,x,\omega) = 1_{]a,\,b]}(t)\; 1_A(x)\; X(\omega), \text{ where } 0 \le a \le b, \quad A \in \mathcal{B}_b(\mathbb{R}^d) \text{ and } X \text{ is a bounded } \mathcal{G}^0_a\text{-measurable random variable} \Big\}.$$

For an element of $\mathcal{E}^0$, its stochastic integral with respect to the martingale measure M^0 is defined by

$$(\phi \cdot M^0)_t(B) = X\; (M^0_{t \wedge b}(A \cap B) - M^0_{t \wedge a}(A \cap B)), \qquad t \in \mathbb{R}_+,\; B \in \mathcal{B}(\mathbb{R}^d).$$

We have the following isometry:

$$\mathbb{E}((\phi \cdot M^0)_t(B)\; \overline{(\psi \cdot M^0)_t(C)}) = \langle \phi\, 1_B, \psi\, 1_C \rangle_{t,0}, \qquad \forall \phi, \psi \in \mathcal{E}^0,\; B, C \in \mathcal{B}(\mathbb{R}^d), \tag{5.10}$$

where

$$\langle \phi\, 1_B, \psi\, 1_C \rangle_{t,0} = \mathbb{E}\left(\int_0^t ds \int_{B \times C} \Gamma_0(dx, dy)\; \phi(s,x)\; \overline{\psi(s,y)} \right). \tag{5.11}$$

Let us moreover denote by $\| \cdot \|_{t,0}$ the semi-norm induced by the semi-scalar product $\langle \cdot, \cdot \rangle_{t,0}$.

We extend now the stochastic integral $(\phi \cdot M^0)_t(B)$ to more general integrands. Let us first consider linear combinations of elementary integrands. For

$$\phi = \sum_{i=1}^m c_i\; \phi_i, \qquad \text{where } n \ge 1,\; c_1, \ldots, c_m \in \mathbb{C},\; \phi_1, \ldots, \phi_m \in \mathcal{E}^0,$$

we define

$$(\phi \cdot M^0)_t(B) = \sum_{i=1}^m c_i\; (\phi_i \cdot M^0)_t(B), \qquad t \in \mathbb{R}_+,\; B \in \mathcal{B}(\mathbb{R}^d).$$

One can check that this definition is correct, since it does not depend on the decomposition chosen for ϕ. Moreover, the stochastic integral is a random linear functional in ϕ (in the sense of (5.4)) and the isometry property (5.10) remains satisfied. Let then $\mathcal{P}^0$ be the *predictable σ-field* generated by the functions of $\mathcal{E}^0$, and term *predictable functions* the functions which are

$\mathcal{P}^0$-measurable (note that any Borel-measurable function $\phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$ is a *deterministic* predictable function). For $t \in \mathbb{R}_+$ and predictable $\phi : [0,t] \times \mathbb{R}^d \times \Omega \to \mathbb{C}$, let us define

$$\|\phi\|_{t,+,0}^2 = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^d \times \mathbb{R}^d} \nu_0(dx,dy)\, |\phi(s,x)\ \phi(s,y)| \right).$$

Moreover, set

$$H_{t,+,0} = \left\{ \phi : [0,t] \times \mathbb{R}^d \times \Omega \to \mathbb{C} \;\middle|\; \phi \text{ is predictable and } \|\phi\|_{t,+,0} < \infty \right\}.$$

By classical arguments (see [62, Chap. 2]), the stochastic integral $(\phi \cdot M^0)_t(B)$ can be extended to elements of $H_{t,+,0}$. Furthermore, both the a.s. linearity and the isometry property (5.10) remain satisfied. In the following, we will adopt the notation $(\phi \cdot M^0)_t = (\phi \cdot M^0)_t(\mathbb{R}^d)$.

Note also that the stochastic integral $(\phi \cdot M^0)_t$ of a deterministic integrand, being the limit in $L^2(\Omega)$ of Gaussian variables, is itself Gaussian, and that in a similar manner, the process $(\phi \cdot M^0) = \{(\phi \cdot M^0)_t,\ t \in \mathbb{R}_+\}$ is a Gaussian process. Furthermore, we have the following isometry for deterministic integrands ϕ and ψ:

$$\mathbb{E}((\phi \cdot M^0)_t\ \overline{(\psi \cdot M^0)_t}) = \int_0^t ds \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx,dy)\ \phi(s,x)\ \overline{\psi(s,y)}. \tag{5.12}$$

The following stochastic Fubini theorem will be used to show the existence of a solution to equation (5.9); a similar theorem can be found in [62, Chap. 2]. The present version is given here only for continuous deterministic integrands, since this is all we need in the following. For $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$ and $s \in \mathbb{R}_+$, let us define $\phi_s(r,x) = \phi(s,r,x)$, $r \in \mathbb{R}_+$, $x \in \mathbb{R}^d$.

Theorem 5.3.1. *If $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$ is continuous, $t \in \mathbb{R}_+$ and*

$$\int_0^t ds \int_0^t dr \int_{\mathbb{R}^d \times \mathbb{R}^d} \nu_0(dx,dy)\ |\phi(s,r,x)\ \phi(s,r,y)| < \infty,$$

then

$$\int_0^t ds\ (\phi_s \cdot M^0)_t = (\psi_t \cdot M^0)_t, \qquad \mathbb{P}-a.s., \tag{5.13}$$

where $\psi(t,r,x) = \displaystyle\int_0^t ds\ \phi(s,r,x).$

Proof. Let us introduce the following notation:

$$\langle \phi_1, \phi_2 \rangle_{t,+,0} = \int_0^t ds \int_{\mathbb{R}^d \times \mathbb{R}^d} \nu_0(dx,dy)\ \phi_1(s,x)\ \overline{\phi_2(s,y)},$$

Since ν_0 is non-negative definite, this is a semi-scalar product and the following Cauchy-Schwarz inequality is satisfied:

$$\langle \phi_1, \phi_2 \rangle_{t,+,0} \le \|\phi_1\|_{t,+,0}\ \|\phi_2\|_{t,+,0}. \tag{5.14}$$

Let us then show that ψ_t belongs to $H_{t,+,0}$: it is clearly continuous since ϕ is, and

$$\begin{aligned}
\|\psi_t\|^2_{t,+,0} &= \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_0(dx,dy)\ |\psi_t(s,x)|\ |\psi_t(s,y)| \\
&\le \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_0(dx,dy) \int_0^t dr \int_0^t dq\ |\phi(r,s,x)|\ |\phi(q,s,y)| \\
&= \int_0^t dr \int_0^t dq\ \langle|\phi_r|,|\phi_q|\rangle_{t,+,0} \\
&\le \left(\int_0^t dr\ \|\phi_r\|_{t,+,0}\right)^2 \\
&\le t \int_0^t dr\ \|\phi_r\|^2_{t,+,0} \\
&= t \int_0^t dr \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_0(dx,dy)\ |\phi(r,s,x)|\ |\phi(r,s,y)| < \infty.
\end{aligned}$$

In order to show that both sides of (5.13) are equal $\mathbb{P}-a.s.$, let us proceed as in the proof of Theorem 2.5.2 and compute

$$\begin{aligned}
\mathbb{E}\left(\left|\int_0^t dr\ (\phi_r\cdot M^0)_t\right|^2\right) &= \int_0^t dr \int_0^t dq\ \mathbb{E}\left((\phi_r\cdot M^0)_t\ \overline{(\phi_q\cdot M^0)_t}\right) \\
&= \int_0^t dr \int_0^t dq \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\ \phi(r,s,x)\ \overline{\phi(q,s,y)} \\
&= \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\ \psi(t,s,x)\ \overline{\psi(t,s,y)} \\
&= \mathbb{E}\left(\left|(\psi_t\cdot M^0)_t\right|^2\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left(\int_0^t dr\ (\phi_r\cdot M^0)_t\ \overline{(\psi_t\cdot M^0)_t}\right) &= \int_0^t dr\ \mathbb{E}\left((\phi_r\cdot M^0)_t\ \overline{(\psi_t\cdot M^0)_t}\right) \\
&= \int_0^t dr \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\ \psi(t,s,y)\ \overline{\phi(r,s,y)} \\
&= \int_0^t ds \int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\ \psi(t,s,y)\ \overline{\psi(t,s,y)} \\
&= \mathbb{E}\left(\left|(\psi_t\cdot M^0)_t\right|^2\right).
\end{aligned}$$

Since this covariance is equal to the two variances above, the proof is complete. □

We will use the following special case of the preceding theorem.

Corollary 5.3.2. *Let $D=\{(s,r,x)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d : s\ge r\}$ (for $\phi : D\to\mathbb{C}$, $s\in\mathbb{R}_+$, note that ϕ_s is defined on $[0,s]\times\mathbb{R}^d$). If $\phi : D\to\mathbb{C}$ is Borel-measurable, $t\in\mathbb{R}_+$ and*

$$\int_0^t ds \int_0^s dr \int_{\mathbb{R}^d\times\mathbb{R}^d} \nu_0(dx,dy)\ |\phi(s,r,x)\ \phi(s,r,y)| < \infty,$$

then

$$\int_0^t ds\ (\phi_s \cdot M^0)_s = (\psi_t \cdot M^0)_t, \qquad \mathbb{P}-a.s.,$$

where $\psi(t,r,x) = \int_r^t ds\ \phi(s,r,x)$.

Proof. Replace $\phi(s,r,x)$ by $\phi(s,r,x)\cdot 1_{s\geq r}$ in Theorem 5.3.1. □

5.4 Properties of the Green kernel

Let G be the solution of

$$\frac{\partial^2 G}{\partial t^2} + 2a\,\frac{\partial G}{\partial t} + b\,G - \Delta G = 0, \quad G(0) = 0, \quad \frac{\partial G}{\partial t}(0) = \delta_0. \tag{5.15}$$

G is called the *Green kernel* of equation (5.1). We need to study carefully its properties before studying equation (5.1). Note that in the following, the dependence on a or b of any function (like the Green kernel or a "constant") will be omitted in order to simplify the notation.

From the explicit expressions of G listed in Appendix A, we deduce that for $d \leq 3$ and arbitrary a, b, or arbitrary $d \geq 1$ and $a = b = 0$, G satisfies the following property: for all $t \in \mathbb{R}_+$, $G(t,\cdot)$ is a finite order distribution with compact support on $\mathbb{R}^d$ and there exist $K(t) > 0$ and $N \in \mathbb{N}$ such that

$$\sup_{s\in[0,t]} |G(s,\varphi)| \leq K(t) \sum_{|\underline{n}|\leq N}\ \sup_{x\in\overline{B(0,t)}} |\partial^{\underline{n}}\varphi(x)|, \qquad \forall\varphi\in\mathcal{S}(\mathbb{R}^d), \tag{5.16}$$

where $\underline{n} = (n_1,\ldots,n_d)$ denotes a multi-index in $\mathbb{N}^d$ and $|\underline{n}| = n_1 + \cdots + n_d$.

On the other hand, we deduce easily from (5.15) that the Fourier transform of G in x satisfies

$$\begin{cases} \dfrac{\partial^2 \mathcal{F}G}{\partial t^2}(t,\xi) + 2a\,\dfrac{\partial \mathcal{F}G}{\partial t}(t,\xi) + (b+|\xi|^2)\ \mathcal{F}G(t,\xi) = 0, & t\in\mathbb{R},\ \xi\in\mathbb{R}^d, \\ \mathcal{F}G(0,\xi) = 0, \quad \dfrac{\partial \mathcal{F}G}{\partial t}(0,\xi) = 1, & \xi\in\mathbb{R}^d. \end{cases} \tag{5.17}$$

Solving this ordinary differential equation in t gives the following expression for $\mathcal{F}G$, which is valid for *every* positive natural number d:

$$\mathcal{F}G(t,\xi) = \begin{cases} e^{-at}\,\dfrac{\sin\left(t\sqrt{|\xi|^2+b-a^2}\right)}{\sqrt{|\xi|^2+b-a^2}}, & \text{if} \quad |\xi|^2 > a^2-b, \\ e^{-at}\ t, & \text{if} \quad a^2-b\geq 0 \quad \text{and} \quad |\xi|^2 = a^2-b, \\ e^{-at}\,\dfrac{\sinh\left(t\sqrt{a^2-b-|\xi|^2}\right)}{\sqrt{a^2-b-|\xi|^2}}, & \text{if} \quad a^2-b>0 \quad \text{and} \quad |\xi|^2 < a^2-b. \end{cases} \tag{5.18}$$

Note that, as for (2.6), the first of these three expressions contains the other two. From these, we also deduce that $\mathcal{F}G$ is a real-valued and continuous function on $\mathbb{R}\times\mathbb{R}^d$, which is symmetric and infinitely differentiable in ξ, since for all $t \in \mathbb{R}$, $\mathcal{F}G(t, \cdot)$ is an analytic function on $\mathbb{R}^d$, whose Taylor series is given by

$$\mathcal{F}G(t,\xi) = e^{-at} \sum_{n \in \mathbb{N}} (-1)^n \frac{t^{2n+1}}{(2n+1)!} (|\xi|^2 + b - a^2)^n, \qquad \forall \xi \in \mathbb{R}^d,$$

and since $\mathcal{F}G(t, \cdot)$ and all its derivatives vanish at infinity, $\mathcal{F}G(t, \cdot) \in O_M(\mathbb{R}^d)$. Moreover, $\mathcal{F}G$ satisfies the following properties: for all $T > 0$, there exits $C_0(T) > 0$ such that

$$|\mathcal{F}G(t,\xi)| \leq C_0(T), \qquad \forall t \in [0,T],\ \xi \in \mathbb{R}^d, \tag{5.19}$$

and

$$|\mathcal{F}G(t,\xi) - \mathcal{F}G(s,\xi)| \leq C_0(T)\ |t-s|, \qquad \forall s,t \in [0,T],\ \xi \in \mathbb{R}^d. \tag{5.20}$$

Furthermore, we have the following estimates, which will be used in the next chapter.

Lemma 5.4.1. *For all $t > 0$, there exists $C_1(t) > 0$ such that*

$$\mathcal{F}G(s,\xi)^2 \leq \frac{C_1(t)}{1+|\xi|^2}, \qquad \forall s \in [0,t],\ \xi \in \mathbb{R}^d.$$

Proof. If $|\xi|^2 \geq 2(a^2 - b) + 1$, then (2.10) implies that we obtain that

$$|\xi|^2 + b - a^2 \geq \frac{1+|\xi|^2}{2}, \tag{5.21}$$

so we have

$$\mathcal{F}G(s,\xi)^2 = e^{-2as} \frac{\sin^2\left(s\sqrt{|\xi|^2 + b - a^2}\right)}{|\xi|^2 + b - a^2} \leq e^{2a^- t} \frac{2}{1+|\xi|^2}.$$

If $2(a^2 - b) + 1 \geq 0$ and $a^2 - b \leq |\xi|^2 \leq 2(a^2 - b) + 1$, then

$$\mathcal{F}G(s,\xi)^2 = e^{-2as}\ s^2 \frac{\sin^2\left(s\sqrt{|\xi|^2 + b - a^2}\right)}{s^2\ (|\xi|^2 + b - a^2)} \leq e^{2a^- t}\ t^2,$$

since $r^{-2}\ \sin^2(r) \leq 1$ for all $r \geq 0$. Finally, if $a^2 - b \geq 0$ and $|\xi|^2 \leq a^2 - b$, then

$$\mathcal{F}G(s,\xi)^2 = e^{-2as}\ s^2 \frac{\sinh^2\left(s\sqrt{a^2 - b - |\xi|^2}\right)}{s^2\ (a^2 - b - |\xi|^2)} \leq e^{2a^- t}\ t^2\ \cosh(\sqrt{a^2-b}\, t)^2,$$

since $r^{-2}\ \sinh^2(r) \leq \cosh(r)^2$ for all $r \geq 0$ and cosh is an increasing function. Summing up these estimates, we obtain finally that there exist $R, K_1(t), K_2(t) > 0$ such that

$$\mathcal{F}G(s,\xi)^2 \leq \frac{K_1(t)}{1+|\xi|^2}, \qquad \forall s \in [0,t],\ |\xi| \geq R,$$

and

$$\mathcal{F}G(s,\xi)^2 \leq K_2(t) \leq K_2(t)\ \frac{1+R^2}{1+|\xi|^2}, \qquad \forall s \in [0,t],\ |\xi| \leq R.$$

Defining $C_1(t) = \max(K_1(t), K_2(t)\ (1+R^2))$ gives the desired result. □

The above lemma will be used for rather technical purposes; as a consequence, we can directly obtain the following upper bound, which we list separately for later reference, and which is also satisfied by the Green kernel of the heat equation, while Lemma 5.4.1 is not.

Lemma 5.4.2. *For all $t > 0$, there exists $C_2(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \le \frac{C_2(t)}{1+|\xi|^2}, \qquad \forall \xi \in \mathbb{R}^d.$$

Proof. We obtain this inequality by a simple integration in s of the result of Lemma 5.4.1. □

The following lemma gives a corresponding lower bound.

Lemma 5.4.3. *For all $t > 0$, there exists $C_3(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \ge \frac{C_3(t)}{1+|\xi|^2}, \qquad \forall \xi \in \mathbb{R}^d.$$

Proof. If $|\xi|^2 \ge a^2 - b + \frac{1}{t^2}$, then (2.11) implies that

$$|\xi|^2 + b - a^2 \le (1 \vee (b-a^2))\ (1+|\xi|^2), \tag{5.22}$$

so we obtain that

$$\begin{aligned}\int_0^t ds\ \mathcal{F}G(s,\xi)^2 &= \int_0^t ds\ e^{-2as}\ \frac{\sin^2\left(s\sqrt{|\xi|^2+b-a^2}\right)}{|\xi|^2+b-a^2} \\ &\ge \frac{e^{-2a^+t}}{1\vee(b-a^2)}\ \frac{1}{1+|\xi|^2} \int_0^t ds\ \sin^2\left(s\sqrt{|\xi|^2+b-a^2}\right).\end{aligned}$$

This implies that

$$\int_0^t ds\ \sin^2\left(s\sqrt{|\xi|^2+b-a^2}\right) = \frac{t}{2}\left(1 - \frac{\sin\left(2t\sqrt{|\xi|^2+b-a^2}\right)}{2t\sqrt{|\xi|^2+b-a^2}}\right) \ge \frac{t}{4},$$

since $t\sqrt{|\xi|^2+b-a^2} \ge 1$ and $|\sin(r)| \le 1$ for all $r \ge 0$. So we obtain finally that

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \ge \frac{e^{-2a^+t}}{(1\vee(b-a^2))}\ \frac{t}{4}\ \frac{1}{1+|\xi|^2}$$

If $a^2 - b + \frac{1}{t^2} \ge 0$ and $a^2 - b \le |\xi|^2 \le a^2 - b + \frac{1}{t^2}$, then

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ e^{-2as}\ s^2\ \frac{\sin^2\left(s\sqrt{|\xi|^2+b-a^2}\right)}{s^2\ (|\xi|^2+b-a^2)} \ge e^{-2a^+t}\ \frac{t^3}{3}\ \sin^2(1),$$

since $s\sqrt{|\xi|^2+b-a^2} \le 1$ for all $s \in [0,t]$ and $r^{-2}\ \sin^2(r) \ge \sin(1)^2 > 0$ for all $r \in [0,1]$. Finally, if $a^2 - b \ge 0$ and $|\xi|^2 \le a^2 - b$, then

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ e^{-2as}\ s^2\ \frac{\sinh^2\left(s\sqrt{a^2-b-|\xi|^2}\right)}{s^2\ (a^2-b-|\xi|^2)} \ge e^{-2a^+t}\ \frac{t^3}{3},$$

since $r^{-2}\ \sinh^2(r) \geq 1$ for all $r \geq 0$. Summing up these estimates, we obtain finally that there exist $R(t), K_1(t), K_2(t) > 0$ such that

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \geq \frac{K_1(t)}{1+|\xi|^2}, \qquad \forall |\xi| \geq R(t),$$

and

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \geq K_2(t) \geq \frac{K_2(t)}{1+|\xi|^2}, \qquad \forall |\xi| \leq R(t).$$

Defining $C_3(t) = \min(K_1(t), K_2(t))$ gives the desired result. □

As for the linear hyperbolic equation in a bounded domain, let us also define $H = \frac{\partial G}{\partial t} + 2a\ G$. We easily see that H satisfies

$$\frac{\partial^2 H}{\partial t^2} + 2a\ \frac{\partial H}{\partial t} + b\ H - \Delta H = 0, \quad H(0) = \delta_0, \quad \frac{\partial H}{\partial t}(0) = 0. \tag{5.23}$$

The equation follows directly from the definition of H and equation (5.15). In order to check the initial conditions, let us compute

$$\mathcal{F}H(t,\xi) = e^{-at}\ \cos\left(t\sqrt{|\xi|^2+b-a^2}\right) + a\ e^{-at}\ \frac{\sin\left(t\sqrt{|\xi|^2+b-a^2}\right)}{\sqrt{|\xi|^2+b-a^2}}, \tag{5.24}$$

therefore $\mathcal{F}H(0,\xi) = 1$, and

$$\frac{\partial \mathcal{F}H}{\partial t}(t,\xi) = -e^{-at}\ \sqrt{|\xi|^2+b-a^2}\ \sin\left(t\sqrt{|\xi|^2+b-a^2}\right) - a^2\ e^{-at}\ \frac{\sin\left(t\sqrt{|\xi|^2+b-a^2}\right)}{\sqrt{|\xi|^2+b-a^2}},$$

therefore $\frac{\partial \mathcal{F}H}{\partial t}(0,\xi) = 0$.

5.5 Existence and uniqueness of the weak solution

To show the existence of a solution to equation (5.9), we will need the following three lemmas.

Lemma 5.5.1. *For $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the functions $\phi_{t,\varphi}, \psi_{t,\varphi} : [0,t] \to \mathcal{S}(\mathbb{R}^d)$ defined by*

$$\phi_{t,\varphi}(s,x) = (G(t-s) * \varphi)(x) \quad \textit{and} \quad \psi_{t,\varphi}(s,x) = \left(\frac{\partial G}{\partial t}(t-s) * \varphi\right)(x), \qquad s \in [0,t],\ x \in \mathbb{R}^d,$$

belong to $H_{t,+,0}$.

Proof. The argument is the same for $\phi_{t,\varphi}$ and $\psi_{t,\varphi}$. Let us then consider only $\phi_{t,\varphi}$. We first show that the map $(s,x) \mapsto (G(t-s) * \varphi)(x)$ is continuous, therefore Borel-measurable. The continuity in x is a consequence of the fact that for fixed $s \in [0,t]$, the map $x \mapsto (G(t-s)*\varphi)(x)$ belongs to $\mathcal{S}(\mathbb{R}^d)$; it remains then to show that the map $s \mapsto (G(t-s) * \varphi)(x)$ is continuous on $[0,t]$, uniformly in $x \in \mathbb{R}^d$. Using the identity $\mathcal{F}^{-1} \circ \mathcal{F} = Id$, we obtain by (4.1) and (4.3) that

$$\begin{aligned}
&|(G(t-(s+h)) * \varphi)(x) - (G(t-s) * \varphi)(x)| \\
&= \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi\ (\mathcal{F}G(t-s-h,\xi) - \mathcal{F}G(t-s,\xi))\ \mathcal{F}\varphi(\xi)\ \chi_{-x}(\xi) \right| \\
&\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi\ |\mathcal{F}G(t-s-h,\xi) - \mathcal{F}G(t-s,\xi)|\ |\mathcal{F}\varphi(\xi)|.
\end{aligned}$$

Since $\mathcal{F}G$ is continuous by (5.18), $\mathcal{F}\varphi \in \mathcal{S}(\mathbb{R}^d)$ and using (5.20), we can apply the dominated convergence theorem to conclude that the above expression, which does not depend on x, tends to 0 as $h \to 0$, so the continuity in s is proven and it is uniform in x.

Secondly, for all $s \in [0,t]$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \nu_0(dx,dy)\, |\phi_{t,\varphi}(s,x)\ \phi_{t,\varphi}(s,y)| < \infty,$$

since $\phi_{t,\varphi}(s,\cdot) = G(t-s) * \varphi \in \mathcal{S}(\mathbb{R}^d)$ and ν_0 is a tempered measure on $\mathbb{R}^d \times \mathbb{R}^d$. To show that $\|\phi_{t,\varphi}\|_{t,+,0} < \infty$, it suffices then to show that for all $r > 0$, there exists $C > 0$ such that

$$\sup_{s\in[0,t]} |\phi_{t,\varphi}(s,x)| \leq \frac{C}{(1+|x|)^r}, \qquad \forall x \in \mathbb{R}^d. \tag{5.25}$$

By (5.16), we obtain that

$$\sup_{s\in[0,t]} |\phi_{t,\varphi}(s,x)| = \sup_{s\in[0,t]} |G(t-s, \varphi(x-\cdot))| \leq K(t) \sum_{|\underline{n}|\leq N} \sup_{z\in\overline{B(0,t)}} |\partial^{\underline{n}}\varphi(x-z)|.$$

But since $\varphi \in \mathcal{S}(\mathbb{R}^d)$, for all $r > 0$, there exists $K > 0$ such that

$$\sup_{|\underline{n}|\leq N} |\partial^{\underline{n}}\varphi(x)| \leq \frac{K}{(1+|x|)^r}.$$

Using the fact that $|x-z| \geq |x| - t$ for all $z \in \overline{B(0,t)}$, we obtain

$$\sup_{s\in[0,t]} |\phi_{t,\varphi}(s,x)| \leq K(t)\ (N+1)^d\ K\ \left(\frac{1}{(1+|x|-t)^r} \wedge 1\right),$$

so (5.25) is satisfied with $C = K(t)\ (N+1)^d\ K\ 2^r$. This completes the proof. □

Let us now define the process $P = \{P_t(\varphi),\ t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ by

$$P_t(\varphi) = ((G(t-\cdot) * \varphi) \cdot M^0)_t.$$

By the preceding lemma, this process is well defined. Moreover, it will turn out to be the first component u of the solution of equation (5.9), with vanishing initial conditions. It is a centered Gaussian process with the following covariance, which can be easily deduced from (5.12):

$$\mathbb{E}(P_t(\varphi)\ \overline{P_s(\psi)}) = \int_0^{t\wedge s} dr \int_{\mathbb{R}^d\times\mathbb{R}^d} \Gamma_0(dx,dy)\ (G(t-r)*\varphi)(x)\ \overline{(G(s-r)*\psi)(y)}. \tag{5.26}$$

Furthermore, it satisfies the following properties.

Lemma 5.5.2. *For all $t \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$P_t(\lambda\,\varphi + \psi) = \lambda\ P_t(\varphi) + P_t(\psi), \quad \mathbb{P} - a.s.$$

and there exists $n \in \mathbb{N}$ such that for all $T > 0$, there exists $C_T > 0$ such that

$$\mathbb{E}(|P_t(\varphi) - P_s(\varphi)|^2) \leq C_T\ \|\varphi\|_n^2\ |t-s|, \qquad \forall s,t \in [0,T],\ \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $\|\cdot\|_n$ is the Sobolev norm defined by (4.4).

Proof. The first statement follows from the a.s. linearity of the stochastic integral. In order to prove the second one, we use (5.6), which states that there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx, dy)\ \varphi(x)\ \overline{\varphi(y)} \leq C\ \|\varphi\|_n^2.$$

Fix now $T > 0$. By formula (5.26), we have for all $s, t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}(|P_t(\varphi) - P_s(\varphi)|^2) \\
&= \int_0^s dr \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx, dy)\ ((G(t-r) - G(s-r)) * \varphi)(x)\ \overline{((G(t-r) - G(s-r)) * \varphi)(y)} \\
&\quad + \int_s^t dr \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx, dy)\ (G(t-r) * \varphi)(x)\ \overline{(G(t-r) * \varphi)(y)} \\
&\leq C \left(\int_0^s dr\ \|(G(t-r) - G(s-r)) * \varphi\|_n^2 + \int_s^t dr\ \|G(t-r) * \varphi\|_n^2 \right).
\end{aligned}$$

Using now (4.4) and properties (5.19) and (5.20), we obtain that

$$\begin{aligned}
\|(G(t-r) - G(s-r)) * \varphi\|_n^2 &= \int_{\mathbb{R}^d} d\xi\ (1 + |\xi|^2)^n\ |\mathcal{F}G(t-r, \xi) - \mathcal{F}G(t-s, \xi)|^2\ |\mathcal{F}\varphi(\xi)|^2 \\
&\leq C_0(T)^2\ \|\varphi\|_n^2\ |t-s|^2
\end{aligned}$$

and

$$\|G(t-r) * \varphi\|_n^2 = \int_{\mathbb{R}^d} d\xi\ (1 + |\xi|^2)^n\ |\mathcal{F}G(t-r, \xi)|^2\ |\mathcal{F}\varphi(\xi)|^2 \leq C_0(T)\ \|\varphi\|_n^2.$$

This implies finally that

$$\mathbb{E}(|P_t(\varphi) - P_s(\varphi)|^2) \leq C\ T\ C_0(T)^2\ \|\varphi\|_n^2\ |t-s|^2 + C_0(T)\ \|\varphi\|_n^2\ |t-s| \leq C_T\ \|\varphi\|_n^2\ |t-s|,$$

where $C_T = 2\ C\ T^2\ C_0(T)^2\ + C_0(T)$. This completes the proof. □

A direct consequence of this lemma combined with Proposition 5.1.4 is that the process P admits a modification $\hat{P}$ such that $\hat{P}_t(\cdot) \in \mathcal{S}'(\mathbb{R}^d)$ for all $t \in \mathbb{R}_+$ and $\mathbb{P} - a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto \hat{P}_t(\varphi)$ is continuous from $\mathbb{R}_+$ to $\mathbb{C}$. We will implicitely consider this modification in the following.

Note that the entire preceding analysis gives rise to the same conclusions for the process $Q = \{Q_t(\varphi),\ t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ defined by

$$Q_t(\varphi) = \left(\left(\frac{\partial G}{\partial t}(t - \cdot) * \varphi \right) \cdot M^0 \right)_t,$$

which will turn out to be the second component v of the solution of equation (5.9) with vanishing initial conditions.

The last lemma is a classical one concerning the determistic equation.

Lemma 5.5.3. *For $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, $t_0 \in \mathbb{R}$, the functions $p, q : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ defined by*

$$p(t,x) = (H(t-t_0) * \phi)(x) + (G(t-t_0) * \psi)(x)$$

and

$$q(t,x) = \left(\frac{\partial H}{\partial t}(t-t_0) * \phi\right)(x) + \left(\frac{\partial G}{\partial t}(t-t_0) * \psi\right)(x)$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, satisfy the following two equations:

$$\begin{cases} p(t,x) = \phi(x) + \displaystyle\int_{t_0}^t dr\ q(r,x), \\ \\ q(t,x) = \psi(x) + \displaystyle\int_{t_0}^t dr\ (-2a\ q(r,x) - b\ p(r,x) + \Delta p(r,x)), \end{cases} \tag{5.27}$$

Moreover, $p(t,\cdot)$ and $q(t,\cdot)$ belong to $\mathcal{S}(\mathbb{R}^d)$, for all $t \in \mathbb{R}$.

Proof. These equations simply follow from the definition of p and q and equations (5.15) and (5.23). □

We can now state the existence and uniqueness theorem.

Theorem 5.5.4. *Let $(u_0, u_1) \in \mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d)$ and define $v_0 = u_1 + 2a\ u_0$. The process $(u,v) = \{(u(t), v(t)),\ t \in \mathbb{R}_+\}$ with values in $\mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d)$ defined by*

$$u(t) = u^0(t) + P_t \quad \textit{and} \quad v(t) = v^0(t) + Q_t, \tag{5.28}$$

where

$$\begin{cases} u^0(t) = H(t) * u_0 + G(t) * v_0, & P_t(\varphi) = ((G(t-\cdot) * \varphi) \cdot M^0)_t, \\ \\ v^0(t) = \dfrac{\partial H}{\partial t}(t) * u_0 + \dfrac{\partial G}{\partial t}(t) * v_0, & Q_t(\varphi) = \left(\left(\dfrac{\partial G}{\partial t}(t-\cdot) * \varphi\right) \cdot M^0\right)_t, \end{cases}$$

admits a modification $(\hat{u}, \hat{v})$ which is the unique weak solution of equation (5.1).

Proof. Let us first show existence. Using Lemma 5.5.3, we see that for a fixed $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the functions p and q defined by

$$p(t,x) = (u^0(t) * \tilde{\varphi})(x) \quad \text{and} \quad q(t,x) = (v^0(t) * \tilde{\varphi})(x),$$

where $\tilde{\varphi}(x) = \varphi(-x)$, satisfy equation (5.27) with $t_0 = 0$ and initial conditions $\phi(x) = (u_0 * \tilde{\varphi})(x)$ and $\psi(x) = (v_0 * \tilde{\varphi})(x)$. Evaluating this equation in $x = 0$ gives

$$\begin{cases} \langle u^0(t), \varphi \rangle = \langle u_0, \varphi \rangle + \displaystyle\int_0^t ds\ \langle v^0(s), \varphi \rangle, \\ \\ \langle v^0(t), \varphi \rangle = \langle v_0, \varphi \rangle + \displaystyle\int_0^t ds\ (-2a\ \langle v^0(s), \varphi \rangle - b\ \langle u^0(s), \varphi \rangle + \langle u^0(s), \Delta\varphi \rangle). \end{cases} \tag{5.29}$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

On the other hand, fix now $t \in \mathbb{R}_+$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and define

$$\begin{cases} \zeta_{t,\varphi}(s,x) = (G(t-s)*\varphi)(x) - \displaystyle\int_s^t dr \, \left(\frac{\partial G}{\partial t}(r-s)*\varphi\right)(x), \\ \xi_{t,\varphi}(s,x) = \left(\dfrac{\partial G}{\partial t}(r-s)*\varphi\right)(x) - \displaystyle\int_s^t dr \, \Bigg(-2a \, \left(\frac{\partial G}{\partial t}(r-s)*\varphi\right)(x) \\ \qquad\qquad -b\,(G(r-s)*\varphi)(x) + (G(r-s)*\Delta\varphi)(x)\Bigg) + \varphi(x), \end{cases}$$

Using the fact that for fixed $s \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the functions p and q defined by

$$p(t,x) = (G(t-s)*\varphi)(x) \quad \text{and} \quad q(t,x) = \left(\frac{\partial G}{\partial t}(t-s)*\varphi\right)(x), \qquad t \geq s, \; x \in \mathbb{R}^d,$$

satisfy equation (5.27) with $t_0 = s$, $\phi \equiv 0$, and $\psi = \varphi$, we obtain that $\zeta_{t,\varphi} \equiv \xi_{t,\varphi} \equiv 0$. Moreover, by Lemma 5.5.1 (slightly adapted for the integral terms), all the components of $\zeta_{t,\varphi}$ and $\xi_{t,\varphi}$ belong to $H_{t,+,0}$. So we can write that $(\zeta_{t,\varphi} \cdot M^0)_t = 0$ and $(\xi_{t,\varphi} \cdot M^0)_t = 0$, which gives, by the linearity of the stochastic integral,

$$\begin{cases} ((G(t-\cdot)*\varphi)\cdot M^0)_t = \left(\left(\displaystyle\int_\cdot^t dr \, \frac{\partial G}{\partial t}(r-\cdot)*\varphi\right)\cdot M^0\right)_t, \\ \left(\left(\dfrac{\partial G}{\partial t}(t-\cdot)*\varphi\right)\cdot M^0\right)_t = -2a \, \left(\left(\displaystyle\int_\cdot^t dr \, \frac{\partial G}{\partial t}(r-\cdot)*\varphi\right)\cdot M^0\right)_t \\ \qquad -b\,\left(\left(\displaystyle\int_\cdot^t dr \, G(r-\cdot)*\varphi\right)\cdot M^0\right)_t + \left(\left(\displaystyle\int_\cdot^t dr \, G(r-\cdot)*\Delta\varphi\right)\cdot M^0\right)_t + F_t^0(\varphi). \end{cases}$$

Applying then Corollary 5.3.2 to each integral term leads to the conclusion that the processes P and Q defined in the theorem satisfy the following equation:

$$\begin{cases} P_t(\varphi) = \displaystyle\int_0^t ds \, Q_s(\varphi), \\ Q_t(\varphi) = \displaystyle\int_0^t ds \, (-2a\,Q_s(\varphi) - b\,P_s(\varphi) + P_s(\Delta\varphi)) + F_t^0(\varphi), \end{cases} \tag{5.30}$$

$\mathbb{P}-a.s$, for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Using now Proposition 5.1.4 for all the terms of the above equation (which are shown to satisfy (5.4) and (5.5) by the same arguments as those in Lemma 5.5.2), we obtain that there exist modifications $\hat{P}$ and $\hat{Q}$ of P and Q which satisfy the above equation for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathbb{P}-a.s.$ Combining finally equations (5.29) and (5.30) shows that the process $(u,v) = \{(u(t),v(t)), t \in \mathbb{R}_+\}$ defined by (5.28) admits a modification $(\hat{u},\hat{v})$ such that $\mathbb{P}-a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto (\langle \hat{u}(t),\varphi\rangle, \langle \hat{v}(t),\varphi\rangle)$ is continuous and solves equation (5.9).

In order to prove uniqueness, we follow a classical deterministic argument. Let $(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$ be two solutions of equation (5.9) and define $(\bar{u}, \bar{v}) = (u^{(1)} - u^{(2)}, v^{(1)} - v^{(2)})$. The process $(\bar{u}, \bar{v})$ then satisfies the following equation:

$$\begin{cases} \langle \bar{u}(t), \varphi \rangle = \displaystyle\int_0^t ds\ \langle \bar{v}(s), \varphi \rangle, \\ \\ \langle \bar{v}(t), \varphi \rangle = \displaystyle\int_0^t ds\ (-2a\ \langle \bar{v}(s), \varphi \rangle - b\ \langle \bar{u}(s), \varphi \rangle + \langle \bar{u}(s), \Delta\varphi \rangle). \end{cases} \tag{5.31}$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathbb{P} - a.s.$ Let now p and q satisfy equation (5.27) with $t_0 = T \in \mathbb{R}_+$ and ϕ, ψ arbitrary. Set moreover

$$A(t) = \langle \bar{u}(t), q(t) \rangle - \langle \bar{v}(t), p(t) \rangle.$$

Combining equations (5.27) and (5.31) gives for $t_1, t_2 \in \mathbb{R}_+$,

$$\begin{aligned} A(t_2) - A(t_1) &= \int_{t_1}^{t_2} ds\ (\langle \bar{u}(s), -2a\ q(s) - b\ p(s) + \Delta p(s) \rangle + \langle \bar{v}(s), q(s) \rangle) \\ &\quad - \int_{t_1}^{t_2} ds\ (\langle \bar{v}(s), q(s) \rangle - 2a\ \langle \bar{v}(s), p(s) \rangle - b\ \langle \bar{u}(s), p(s) \rangle + \langle \bar{u}(s), \Delta p(s) \rangle) \\ &= -2a \int_{t_1}^{t_2} ds\ (\langle \bar{u}(s), q(s) \rangle - \langle \bar{v}(s), p(s) \rangle) \\ &= -2a \int_{t_1}^{t_2} ds\ A(s). \end{aligned}$$

Therefore,

$$A(T) = e^{-2aT}\ A(0) = e^{-2aT}\ (\langle \bar{u}(0), q(0) \rangle - \langle \bar{v}(0), p(0) \rangle = 0,$$

since $\bar{u}(0) = \bar{v}(0) = 0$. Using now the terminal conditions $p(T) = \phi$ and $q(T) = \psi$ and considering successively the cases $(\phi, \psi) = (\varphi, 0)$ and $(\phi, \psi) = (0, \varphi)$, we obtain that

$$\langle \bar{u}(T), \varphi \rangle = \langle \bar{v}(T), \varphi \rangle = 0,$$

for arbitrary $T \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathbb{P} - a.s.$, so the conclusion follows. □

Chapter 6

Noise on a k-plane

Let d be a natural number greater than 1 and fix $k \in \{1, ..., d-1\}$. Let us also introduce the following notations.

- For $x \in \mathbb{R}^d \cong \mathbb{R}^k \times \mathbb{R}^{d-k}$, write $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^{d-k}$.

- For $r > 0$ and $a \in \mathbb{R}^k$, let $B_1(a,r)$ denote the ball of center a and radius r in $\mathbb{R}^k$.

- For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, let $\mathcal{F}_1\varphi$ (resp. $\mathcal{F}_2\varphi$) denote the Fourier transform of φ in the coordinates parallel to the k-plane $\mathbb{R}^k \times \{0\}$ (resp. in the perpendicular ones): these are defined by

$$\mathcal{F}_1\varphi(\xi_1, x_2) = \int_{\mathbb{R}^k} dx_1\ \varphi(x_1, x_2)\ \chi_{\xi_1}(x_1)$$

and

$$\mathcal{F}_2\varphi(x_1, \xi_2) = \int_{\mathbb{R}^{d-k}} dx_2\ \varphi(x_1, x_2)\ \chi_{\xi_2}(x_2),$$

where we recall that $\chi_{\xi_i}(x_i) = e^{i\xi_i \cdot x_i}$. These Fourier transforms extend to $T \in \mathcal{S}'(\mathbb{R}^d)$ and note that $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2 = \mathcal{F}_2 \circ \mathcal{F}_1$.

The aim of this chapter and the next one is to study the regularity of the weak solution of equation (5.1) when the measure Γ_0 is formally given by

$$\Gamma_0(x,y) = \Gamma(x_1 - y_1)\ \delta_0(x_2)\ \delta_0(y_2),$$

which can be rigorously written as

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_0(dx, dy)\ \phi(x)\ \psi(y) = \int_{\mathbb{R}^k} \Gamma(dz_1)\ (\phi(\cdot, 0) *_1 \tilde{\psi}(\cdot, 0))(z_1), \qquad \forall \phi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{6.1}$$

where $*_1$ denotes the convolution product in $\mathbb{R}^k$ and $\tilde{\varphi}(x_1) = \overline{\varphi(-x_1)}$ for $x_1 \in \mathbb{R}^k$. This situation corresponds to a noise concentrated on the k-plane $\mathbb{R}^k \times \{0\}$ and spatially homogeneous on this

k-plane. The "classical" equation corresponding to equation (5.9) is then

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) + b\,u(t,x) - \Delta u(t,x) = \dot{F}(t,x_1)\,\delta_0(x_2), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in \mathbb{R}^d. \end{cases} \tag{6.2}$$

Our aim here is to relate the regularity of the weak solution u of this equation to explicit conditions on the covariance of the noise defined above. By the nature of the noise itself, the behavior of the solution will certainly be different along the directions parallel to the k-plane (x_1) and along the perpendicular ones (x_2). Since the noise is spatially homogenous in the coordinate x_1, one should not expect a solution with an L^2-type behavior in x_1 (unless we consider some weighted L^2-space as in [51]). On the contrary, this seems quite plausible for the coordinate x_2.

To be precise, what we are going to show in this chapter is that for $\beta < 1 - \frac{d-k}{2}$ and under some optimal condition B_β on the the spectral measure of the noise (which can be reformulated afterwards into a condition on the covariance: see section 6.6), the weak solution of (6.2) is a process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ indexed by the time variable and the coordinates of the k-plane, with values in some fractional Sobolev space $H^\beta(\mathbb{R}^{d-k})$ (see Section 4.2 for a definition of this space).

This analysis prepares for the study of the following question: when is the weak solution u of equation (6.2) a real-valued process, and not a distribution-valued one? We will see in Chapter 7 that in the case of a noise concentrated on a hyperplane (that is, when $k = d-1$), the optimal condition B_0 obtained here for the process U to be $L^2(\mathbb{R})$-valued is also the optimal condition for the weak solution u of (6.2) to be a real-valued process outside the hyperplane $x_2 = 0$. What we shall also observe in that chapter is that the solution of equation (5.9) cannot be a real-valued process in the case where $k = d-2$, in concordance with the result obtained here that in this case, the process U, if it exists, has to take its values in some fractional Sobolev space $H^\beta(\mathbb{R}^{d-k})$ with β strictly negative.

6.1 Chapter 5 revisited

Though the general results of the preceding chapter apply directly to the particular noise considered here, we prefer to rephrase them somewhat, in order to simplify the analysis later on.

Let us first make precise the assumptions made on the covariance Γ: Γ is assumed to be a signed Borel measure on $\mathbb{R}^k$, which is *non-negative definite on* $\mathbb{R}^k$ in the sense of (4.8), that is,

$$\int_{\mathbb{R}^k} \Gamma(dz_1)\ (\varphi *_1 \tilde{\varphi})(z_1) \geq 0, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^k). \tag{6.3}$$

This implies that Γ is hermitian (cf. [57, Chap. VII, Thm XVII]), hence symmetric, since it is a real-valued measure. Moreover, we assume that there exists a tempered non-negative Borel measure ν on $\mathbb{R}^k$ which is non-negative definite and which dominates $|\Gamma|$, that is,

$$|\Gamma|(A) \leq \nu(A), \qquad \forall A \in \mathcal{B}_b(\mathbb{R}^k).$$

We give here some examples of covariances that satisfy these conditions. Clearly, when Γ is a *non-negative*, tempered and non-negative definite Borel measure on $\mathbb{R}^d$, then $\nu = \Gamma$ satisfies the required assumptions. This non-negativity assumption was taken as a basic assumption in [15, 54] (in the case of a spatially homogeneous noise on $\mathbb{R}^d$) and will be needed in our case for the analysis of non-linear equations (see Chapter 8). Among this class of covariances, we can consider covariances of the form $\Gamma(dx_1) = f(|x_1|)\ dx_1$, where f is a non-negative continuous function on $]0,\infty[$, and examples of such f are

$$f(r) = \frac{1}{r^\gamma}, \quad \text{where} \quad \gamma \in \,]0,k[.$$

We can also show the following. Let λ be the Lebesgue measure on $\mathbb{R}^k$; if Γ is a non-negative definite tempered Borel measure on $\mathbb{R}^d$ such that there exists $C > 0$ where

$$\Gamma + C\lambda \text{ is a non-negative measure on } \mathbb{R}^k, \tag{6.4}$$

then $\nu = \Gamma + 2C\lambda$ satisfies the required assumptions: ν is non-negative definite, being the convex combination of two non-negative definite measures, and

$$|\Gamma| = |\Gamma + C\lambda - C\lambda| \leq |\Gamma + C\lambda| + |C\lambda| = \Gamma + 2C\lambda = \nu.$$

Note that (6.4) was taken as a basic assumption in [51, 52, 53] (in the case of a spatially homogeneous noise on $\mathbb{R}^d$).

Let us now consider the centered Gaussian process $F = \{F_t(\varphi),\ t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^k)\}$ whose covariance is given by

$$\mathbb{E}(F_t(\varphi)\ \overline{F_s(\psi)}) = (t \wedge s) \int_{\mathbb{R}^k} \Gamma(dz_1)\ (\varphi *_1 \tilde{\psi})(z_1), \quad \forall t,s \in \mathbb{R}_+,\ \varphi,\psi \in \mathcal{S}(\mathbb{R}^k).$$

This process is well defined (see Section 4.3), and (6.1) implies that

$$F_t^0(\phi) \stackrel{d}{=} F_t(\phi(\cdot,0)), \qquad \forall t \in \mathbb{R}_+,\ \phi \in \mathcal{S}(\mathbb{R}^d),$$

where $\stackrel{d}{=}$ stands for equality in distribution. This gives an expression for the last term of equation (5.9). Moreover, since we would like to study the regularity of the solution in relation to the regularity of the noise, we restrict ourselves here to the case where $u_0 = v_0 = 0$, and so equation (5.9) becomes

$$\begin{cases} \langle u(t),\varphi\rangle = \displaystyle\int_0^t ds\ \langle v(s),\varphi\rangle, \\[2ex] \langle v(t),\varphi\rangle = \displaystyle\int_0^t ds\ (-2a\ \langle v(s),\varphi\rangle - b\ \langle u(s),\varphi\rangle + \langle u(s),\Delta\varphi\rangle) + F_t(\varphi(\cdot,0)), \end{cases} \tag{6.5}$$

for all $t > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathbb{P} - a.s.$

Remark 6.1.1. When $k = d - 1$ (that is, when the noise is concentrated on a hyperplane), and as mentioned in Remark 3.2.3 for the case of a noise on a sphere, we could intepret the noise term as a boundary term, and therefore consider that (6.5) is the weak formulation of the following classical equation in the upper half space:

$$\frac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\frac{\partial u}{\partial t}(t,x) + b\,u(t,x) - \Delta u(t,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}_+,$$

with the stochastic boundary condition

$$\frac{\partial u}{\partial x_2}(t, x_1, 0) = \dot{F}(t, x_1).$$

Let us now follow the analysis of Chapter 5 and denote by $\{\mathcal{G}_t\}$ the natural augmented filtration of the noise, $\mathcal{E}$ the space of elementary integrands and $\mathcal{P}$ the predictable σ-field; the noise F extends naturally to a worthy martingale measure M and we can define a corresponding stochastic integral $(\phi \cdot M)_t$ for integrands belonging to

$$\begin{aligned} H_{t,+} \;=\; & \Big\{ \phi : [0,t] \times \mathbb{R}^k \times \Omega \to \mathbb{C} \text{ predictable such that} \\ & \quad \|\phi\|_{t,+}^2 = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^k} \nu(dz_1)\ (|\phi(s,\cdot)| *_1 |\tilde{\phi}(s,\cdot)|)(z_1) \right) < \infty \Big\}, \end{aligned}$$

using the isometry

$$\mathbb{E}((\phi \cdot M)_t\ \overline{(\psi \cdot M)_t}) = \langle \phi, \psi \rangle_t = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^k} \Gamma(dz_1)\ (\phi(s,\cdot) *_1 \tilde{\psi}(s,\cdot))(z_1) \right). \tag{6.6}$$

Let us also denote by $\|\cdot\|_t$ the semi-norm induced by the semi-scalar product $\langle \cdot, \cdot \rangle_t$. The stochastic Fubini Theorem 5.3.1 can be adapted to the present situation, and the rest of the analysis is identical to the one of the preceding chapter.

For clarity, we will adopt the following notation for the stochastic integral of a predictable integrand $\phi : [0,t] \times \mathbb{R}^k \times \mathbb{R}^{d-k} \times \Omega \to \mathbb{C}$ restricted to the k-plane $x_2 = 0$:

$$(\phi(\cdot,\cdot,0) \cdot M)_t = \int_{[0,t]\times\mathbb{R}^k} M(ds, dx_1)\ \phi(s, x_1, 0).$$

In particular, the unique solution u of equation (6.5) will be given by

$$\langle u(t), \varphi \rangle = \int_{[0,t]\times\mathbb{R}^k} M(ds, dx_1)\ (G(t-s) * \varphi)(x_1, 0), \qquad t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{6.7}$$

where G is the solution of equation (5.15), whose properties are listed in Section 5.4.

With these modifications in hand, we can proceed further.

6.2 Extension of the stochastic integral

The first technical step towards the study of the regularity of the solution consists in extending the stochastic integral to distribution-valued integrands, since the processes that will appear in the following will be expressed as stochastic integrals of such integrands.

Following [15], we first consider a more general class of martingale measures, in order to include directly the treatment of non-linear equations in our analysis. Let $Z = \{Z(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ be a real-valued predictable process such that for all $T > 0$,

$$\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^k} \mathbb{E}(Z(t,x_1)^2) < \infty. \tag{6.8}$$

By [62, Chap. 2], $M^Z = \{(Z \cdot M)_t(B),\ \mathcal{G}_t,\ t \in \mathbb{R}_+,\ B \in \mathcal{B}_b(\mathbb{R}^k)\}$ defines also a worthy martingale measure with covariation measure

$$Q^Z([0,t] \times A \times B) = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^k} \Gamma(dz_1)\ ((Z(s,\cdot)\ 1_A) *_1 (\tilde{Z}(s,\cdot)\ \tilde{1}_B))(z_1)\right)$$

and dominating measure

$$K^Z([0,t] \times A \times B) = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^k} \nu(dz_1)\ ((|Z(s,\cdot)|\ 1_A) *_1 (|\tilde{Z}(s,\cdot)|\ \tilde{1}_B))(z_1)\right).$$

This implies that we can define the stochastic integral $(\phi \cdot M^Z)_t$ of a Borel-measurable function $\phi : [0,t] \times \mathbb{R}^k \to \mathbb{C}$ such that

$$\|\phi\|^2_{t,+,Z} = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^k} \nu(dz_1)\ (|\phi(s,\cdot)\ Z(s,\cdot)| *_1 |\tilde{\phi}(s,\cdot)\ \tilde{Z}(s,\cdot)|)(z_1)\right) < \infty,$$

and let us denote by $H_{t,+,Z}$ the space of such (deterministic) integrands. Note that if $\phi \in H_{t,+}$ and ϕ is deterministic, then

$$\|\phi\|^2_{t,+,Z} \leq \sup_{(s,x_1)\in[0,t]\times\mathbb{R}^k} \mathbb{E}(Z(s,x_1)^2)\ \|\phi\|^2_{t,+} < \infty,$$

so $\phi \in H_{t,+,Z}$. Moreover, the following isometry property holds:

$$\mathbb{E}((\phi \cdot M^Z)_t\ \overline{(\psi \cdot M^Z)_t}) = \langle \phi, \psi \rangle_{t,Z},$$

where

$$\langle \phi, \psi \rangle_{t,Z} = \mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^k} \Gamma(dz_1)\ ((\phi(s,\cdot)\ Z(s,\cdot)) *_1 (\tilde{\psi}(s,\cdot)\ \tilde{Z}(s,\cdot)))(z_1)\right). \tag{6.9}$$

Let us also denote by $\|\cdot\|_{t,Z}$ the semi-norm induced by the semi-scalar product $\langle \cdot, \cdot \rangle_{t,Z}$.

We can now proceed to the extension of the stochastic integral. If we assume that Z satisfies

$$\mathbb{E}(Z(t,x_1)\ Z(t,y_1)) = \mathbb{E}(Z(t,0)\ Z(t,x_1-y_1)), \qquad \forall t \in \mathbb{R}_+,\ x_1,\ y_1 \in \mathbb{R}^k, \tag{6.10}$$

then the function $\gamma : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}$ defined by

$$\gamma(t, z_1) = \mathbb{E}(Z(t,0)\ Z(t,z_1)), \qquad (t, z_1) \in \mathbb{R}_+ \times \mathbb{R}^k,$$

is symmetric in z_1 and for $\phi, \psi \in H_{t,+,Z}$, (6.9) can be rewritten as

$$\langle \phi, \psi \rangle_{t,Z} = \int_0^t ds \int_{\mathbb{R}^k} \Gamma(dz_1)\ \gamma(s, z_1)\ (\phi(s,\cdot) *_1 \tilde{\psi}(s,\cdot))(z_1). \tag{6.11}$$

Therefore, for $s \in [0,t]$, the measure Γ_s^Z defined by

$$\Gamma_s^Z(dz_1) = \Gamma(dz_1)\ \gamma(s, z_1)$$

is a non-negative definite measure on $\mathbb{R}^k$, since

$$\int_{\mathbb{R}^k} \Gamma_s^Z(dz_1)\ (\varphi *_1 \tilde{\varphi})(z_1) = \mathbb{E}\left(\int_{\mathbb{R}^k} \Gamma(dz_1)\ (\varphi\ Z(s,\cdot) *_1 \tilde{\varphi}\ \tilde{Z}(s,\cdot))(z_1) \right) \geq 0, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^k),$$

by (6.3). Moreover, since for all $s \in [0,t]$, Γ_s^Z is a signed tempered Borel measure on $\mathbb{R}^k$, it belongs to $\mathcal{S}'(\mathbb{R}^k)$. The Bochner-Schwartz theorem 4.3.1 then implies that there exists a non-negative tempered Borel measure μ_s^Z on $\mathbb{R}^k$ such that $\Gamma_s^Z = \mathcal{F}_1 \mu_s^Z$. Moreover, μ_s^Z is symmetric on $\mathbb{R}^k$ since Γ_s^Z is real-valued. Let us now consider the following subspace of $H_{t,+}$, composed by regular deterministic integrands:

$$H_{t,0} = \Big\{ \phi : [0,t] \times \mathbb{R}^k \to \mathbb{C} \text{ Borel-measurable such that } \|\phi\|_{t,+} < \infty \text{ and } \phi(s,\cdot) \in \mathcal{S}(\mathbb{R}^k), \quad \forall s \in [0,t] \Big\}.$$

If $\phi, \psi \in H_{t,0}$, then $\mathcal{F}_1\phi$, $\mathcal{F}_1\psi$ are Borel-measurable functions and we obtain the following expression for (6.11), using basic properties of the Fourier transform:

$$\langle \phi, \psi \rangle_{t,Z} = \int_0^t ds \int_{\mathbb{R}^k} \mu_s^Z(d\xi_1)\ \mathcal{F}_1\phi(s,\xi_1)\ \overline{\mathcal{F}_1\psi(s,\xi_1)}. \tag{6.12}$$

With this expression in hand, we can finally define a larger space, which contains (deterministic) distribution-valued integrands:

$$H_{t,Z} = \Big\{ \phi : [0,t] \to O'_C(\mathbb{R}^k) \ \Big|\ (s,\xi_1) \mapsto \mathcal{F}_1\phi(s,\xi_1) \text{ is Borel-measurable, } \|\phi\|_{t,Z} < \infty \text{ and } \exists (\phi_n) \subset H_{t,0} \text{ such that } \|\phi - \phi_n\|_{t,Z} \underset{n\to\infty}{\to} 0 \Big\},$$

where $\|\phi\|_{t,Z}$ is *defined* here by

$$\|\phi\|_{t,Z}^2 = \int_0^t ds \int_{\mathbb{R}^k} \mu_s^Z(d\xi_1)\ |\mathcal{F}_1\phi(s,\xi_1)|^2.$$

The stochastic integral $(\phi \cdot M^Z)_t$ extends then by isometry to elements of $H_{t,Z}$. Note that as before, the linearity and the isometry property remain satisfied. We will once again adopt the following notation for the stochastic integral of $\phi : [0,t] \times \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{C}$ restricted to the k-plane $x_2 = 0$:

$$(\phi(\cdot,\cdot,0) \cdot M^Z)_t = \int_{[0,t]\times\mathbb{R}^k} M(ds, dx_1)\; Z(s,x_1)\; \phi(s,x_1,0),$$

even in the case where $\phi(s,\cdot,0)$ is a distribution in x_1.

Note that for the linear equation, we will only need the definition of the stochastic integral when $Z \equiv 1$, in which case we denote the space of integrands by H_t and the isometry (6.6) becomes

$$\mathbb{E}((\phi \cdot M)_t\, \overline{(\psi \cdot M)_t}) = \langle \phi, \psi \rangle_t = \int_0^t ds \int_{\mathbb{R}^k} \mu(d\xi_1)\; \mathcal{F}_1\phi(s,\xi_1)\; \overline{\mathcal{F}_1\psi(s,\xi_1)}. \tag{6.13}$$

Moreover, one can notice that since the integrand considered here is deterministic, the process $(\phi \cdot M) = \{(\phi \cdot M)_t,\ t \in \mathbb{R}_+\}$ is a Gaussian process.

The following theorems will also be useful (cf. [15, Thms 2 and 3] and [16] for proofs). Before stating them, let us denote by $O'_C(\mathbb{R}^k)_+$ the space of non-negative distributions with rapid decrease on $\mathbb{R}^k$.

Theorem 6.2.1. *Let Z be a process satisfying (6.8) and (6.10). If Γ is a non-negative measure on $\mathbb{R}^k$, $\phi : [0,t] \to O'_C(\mathbb{R}^k)_+$ is such that $\mathcal{F}_1\phi$ is a Borel-measurable function and $\|\phi\|_t < \infty$, then $\phi \in H_{t,Z}$ and*

$$\begin{aligned}
\mathbb{E}(|(\phi \cdot M^Z)_t|^2) &= \int_0^t ds \int_{\mathbb{R}^k} \mu_s^Z(d\xi_1)\; |\mathcal{F}_1\phi(s,\xi_1)|^2 \\
&\le \int_0^t ds \sup_{x_1\in\mathbb{R}^k} \mathbb{E}(Z(s,x_1)^2) \int_{\mathbb{R}^k} \mu(d\xi_1)\; |\mathcal{F}_1\phi(s,\xi_1)|^2.
\end{aligned}$$

Theorem 6.2.2. *If $\phi : [0,t] \to O'_C(\mathbb{R}^k)$ is such that $\mathcal{F}_1\phi$ is a Borel-measurable function, $\|\phi\|_t < \infty$ and*

$$\lim_{h\downarrow 0} \int_0^t ds \int_{\mathbb{R}^k} \mu(d\xi_1) \sup_{s<r<s+h} |\mathcal{F}_1\phi(r,\xi_1) - \mathcal{F}_1\phi(s,\xi_1)|^2 = 0, \tag{6.14}$$

then $\phi \in H_t$.

6.3 Fourier transform of the solution in coordinates perpendicular to the k-plane

Let u be the solution of equation (6.5), for which we have an explicit formula given by (6.7). In order to study when this solution is a process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ indexed by the time variable and the coordinates of the k-plane, with values in some fractional Sobolev space

$H^\beta(\mathbb{R}^{d-k})$, we first need to consider the Fourier transform of the solution in the coordinate x_2 perpendicular to the k-plane and see when it is a real-valued process. The reason for this comes from the definition of $H^\beta(\mathbb{R}^{d-k})$, which states (see Section 4.2) that

$$v \in H^\beta(\mathbb{R}^{d-k}) \quad \text{if and only if} \quad \mathcal{F}_2 v \in L^2(\mathbb{R}^{d-k}; (1+|\xi_2|^2)^\beta \, d\xi_2).$$

Therefore, $\mathcal{F}_2 U(t, x_1, \cdot)$ needs at least to be function-valued if one wants $U(t, x_1, \cdot)$ to belong to $H^\beta(\mathbb{R}^{d-k})$.

We will see here that under an explicit condition on the spectral measure μ, which will be shown to be optimal, there exists a real-valued process Y which is the Fourier transform in x_2 of the distribution-valued solution u of equation (6.5), that is,

$$\langle u(t), \mathcal{F}_2\varphi\rangle = \int_{\mathbb{R}^k} dx_1 \int_{\mathbb{R}^{d-k}} d\xi_2 \; Y(t, x_1, \xi_2)\; \varphi(x_1, \xi_2), \qquad \mathbb{P} - a.s., \; \forall t \in \mathbb{R}_+, \; \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(Note that by definition, $\langle u(t), \mathcal{F}_2\varphi\rangle = \langle \mathcal{F}_2 u(t), \varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$). The condition is the following.

Assumption A_0.

$$\int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2} < \infty.$$

Remark 6.3.1. This condition is the same as that obtained for the existence of a real-valued process which is the solution of a hyperbolic equation in $\mathbb{R}^k$ driven by spatially homogeneous noise with spectral measure μ (see [15, 30]). It can be reformulated into an explicit condition on the covariance Γ: see Section 6.6.

In order to show the sufficiency of Assumption A_0, we begin by establishing the following three lemmas.

Lemma 6.3.2. *Under Assumption A_0 and for $(t, x_1, \xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}$, the function $\phi_{t,x_1,\xi_2} : [0,t] \to O'_C(\mathbb{R}^k)$ defined by*

$$\phi_{t,x_1,\xi_2}(s, \cdot) = \mathcal{F}_2 G(t-s, x_1 - \cdot, \xi_2), \qquad s \in [0,t],$$

belongs to H_t.

Before proving this lemma, let us note that the above definition means that

$$\langle \phi_{t,x_1,\xi_2}(s), \varphi\rangle = \langle \mathcal{F}_2 G(t-s, \cdot, \xi_2), \varphi(x_1 - \cdot)\rangle, \qquad s \in [0,t], \; \varphi \in \mathcal{S}(\mathbb{R}^k).$$

Proof. Fix $(t, x_1, \xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}$. It is sufficient to prove that ϕ_{t,x_1,ξ_2} satisfies the conditions of Theorem 6.2.2. First note that for all $s \in [0,t]$, $\xi_1 \in \mathbb{R}^k$,

$$\mathcal{F}_1 \phi_{t,x_1,\xi_2}(s, \xi_1) = \mathcal{F} G(t-s, -\xi_1, \xi_2)\; \chi_{x_1}(\xi_1),$$

so $\mathcal{F}_1\phi_{t,x_1,\xi_2}(s,\cdot) \in O_M(\mathbb{R}^k)$ (see Section 5.4) and this implies by (4.2) that $\phi_{t,x_1,\xi_2}(s,\cdot) \in O'_C(\mathbb{R}^k)$. Moreover, $\mathcal{F}_1\phi_{t,x_1,\xi_2}$ is a continuous and therefore Borel-measurable function, and formula (6.13) gives

$$\begin{aligned}
\|\phi_{t,x_1,\xi_2}\|_t^2 &= \int_0^t ds \int_{\mathbb{R}^k} \mu(d\xi_1)\, \mathcal{F}G(t-s,-\xi_1,\xi_2)^2 \\
&\leq \int_{\mathbb{R}^k} \mu(d\xi_1)\, \frac{C_2(t)}{1+|\xi_1|^2+|\xi_2|^2} \\
&\leq C_2(t) \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2} \\
&< \infty,
\end{aligned}$$

by Lemma 5.4.2 and Assumption A_0. We now check condition (6.14), that is,

$$\lim_{h\downarrow 0} \int_0^t ds \int_{\mathbb{R}^k} \mu(d\xi_1) \sup_{s<r<s+h} |\mathcal{F}_1\phi_{t,x_1,\xi_2}(r,\xi_1) - \mathcal{F}_1\phi_{t,x_1,\xi_2}(s,\xi_1)|^2 = 0.$$

Since $\mathcal{F}G(\cdot,\xi_1,\xi_2)$ is a continuous function, it is uniformly continuous in s on $[0,t]$, so we obtain that for all $(s,\xi_1) \in [0,t] \times \mathbb{R}^k$,

$$\lim_{h\downarrow 0} \sup_{s<r<s+h} |\mathcal{F}G(t-r,-\xi_1,\xi_2)\, \chi_{x_1}(\xi_1) - \mathcal{F}G(t-s,-\xi_1,\xi_2)\, \chi_{x_1}(\xi_1)|^2 = 0.$$

Moreover, by Lemma 5.4.1,

$$\begin{aligned}
&\sup_{s<r<s+h} |\mathcal{F}G(t-r,-\xi_1,\xi_2)\, \chi_{x_1}(\xi_1) - \mathcal{F}G(t-s,-\xi_1,\xi_2)\, \chi_{x_1}(\xi_1)|^2 \\
&\qquad \leq \frac{4\, C_1(t)}{1+|\xi_1|^2+|\xi_2|^2} \leq \frac{4\, C_1(t)}{1+|\xi_1|^2}, \qquad (6.15)
\end{aligned}$$

so we obtain that condition (6.14) is fulfilled, using again Assumption A_0 and the dominated convergence theorem. □

Lemma 6.3.3. *Let M be the worthy martingale measure defined in Section 6.1. Under Assumption A_0, the real-valued process $Y = \{Y(t,x_1,\xi_2),\ (t,x_1,\xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\}$ defined by*

$$Y(t,x_1,\xi_2) = \int_{[0,t]\times\mathbb{R}^k} M(ds,dy_1)\, \mathcal{F}_2G(t-s,x_1-y_1,\xi_2), \qquad (t,x_1,\xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k},$$

is a centered Gaussian process whose covariance is given by

$$\begin{aligned}
&\mathbb{E}(Y(t,x_1,\xi_2)\, Y(s,y_1,\eta_2)) \\
&\quad = \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^{t\wedge s} dr\, \mathcal{F}G(t-r,-\xi_1,\xi_2)\, \mathcal{F}G(s-r,-\xi_1,\eta_2)\, \chi_{x_1-y_1}(\xi_1), \qquad (6.16)
\end{aligned}$$

and is such that the map $(t,x_1,\xi_2) \mapsto Y(t,x_1,\xi_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}$ to $L^2(\Omega)$.

Remark 6.3.4. By [42, Prop. 3.6 and Cor. 3.8], this result implies that the process Y admits a modification $\tilde{Y}$ such that the map $(t,x_1,\xi_2,\omega) \mapsto \tilde{Y}(t,x_1,\xi_2,\omega)$ is jointly measurable. We will implicitely consider this modification in the following.

Proof of Lemma 6.3.3. By Lemma 6.3.2, the process Y is well defined. The fact that it is a centered Gaussian process with the covariance given above follows easily from (6.13) and the remark following it. Moreover, since μ and $\mathcal{F}G$ are symmetric in ξ_1, (6.16) is equal to

$$\int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^{t\wedge s} dr\ \mathcal{F}G(t-r,-\xi_1,\xi_2)\ \mathcal{F}G(s-r,-\xi_1,\eta_2)\ \cos(\xi_1\cdot(x_1-y_1)),$$

and this implies that Y is real-valued.

In order to show that the map $(t,x_1,\xi_2)\mapsto Y(t,x_1,\xi_2)$ is continuous from $\mathbb{R}_+\times\mathbb{R}^k\times\mathbb{R}^{d-k}$ to $L^2(\Omega)$, we will show that for all $T>0$, it is continuous from $[0,T]\times\mathbb{R}^k\times\mathbb{R}^{d-k}$ to $L^2(\Omega)$. We do this in three steps, showing first that the map $\xi_2\mapsto Y(t,x_1,\xi_2)$ is continuous in $L^2(\Omega)$ uniformly in $(t,x_1)\in[0,T]\times\mathbb{R}^k$, then that for fixed $\xi_2\in\mathbb{R}^{d-k}$, the map $x_1\mapsto Y(t,x_1,\xi_2)$ is continuous in $L^2(\Omega)$ uniformly in $t\in[0,T]$ and finally that for fixed $(x_1,\xi_2)\in\mathbb{R}^d$, the map $t\mapsto Y(t,x_1,\xi_2)$ is continuous in $L^2(\Omega)$. These three properties clearly imply joint L^2-continuity of the map $(t,x_1,\xi_2)\mapsto Y(t,x_1,\xi_2)$ on $[0,T]\times\mathbb{R}^k\times\mathbb{R}^{d-k}$.

Let ξ_2, $\eta_2\in\mathbb{R}^{d-k}$. Using (6.16), we obtain that

$$\begin{aligned}&\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^k} \mathbb{E}((Y(t,x_1,\eta_2)-Y(t,x_1,\xi_2))^2)\\ &= \sup_{(t,x_1)\in[0,T]\times\mathbb{R}^k}\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t ds\ (\mathcal{F}G(t-s,-\xi_1,\eta_2)-\mathcal{F}G(t-s,-\xi_1,\xi_2))^2\ |\chi_{x_1}(\xi_1)|^2\\ &= \sup_{t\in[0,T]}\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t dr\ (\mathcal{F}G(r,-\xi_1,\eta_2)-\mathcal{F}G(r,-\xi_1,\xi_2))^2,\end{aligned}$$

where we have used the change of variable $r=t-s$ and the fact that $|\chi_{x_1}(\xi_1)|^2=1$, so the integrand does not depend on x_1 and the supremum over x_1 disappears. Since the integrand is also non-negative, the supremum is attained at $t=T$, so we obtain that

$$\begin{aligned}&\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^k} \mathbb{E}((Y(t,x_1,\eta_2)-Y(t,x_1,\xi_2))^2)\\ &= \int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^T dr\ (\mathcal{F}G(r,-\xi_1,\eta_2)-\mathcal{F}G(r,-\xi_1,\xi_2))^2. \qquad (6.17)\end{aligned}$$

Since $\mathcal{F}G$ is continuous, the integrand in (6.17) converges to 0 as $\eta_2\to\xi_2$. Moreover, by Lemma 5.4.1, we obtain, as in (6.15), that for all $(r,\xi_1)\in[0,T]\times\mathbb{R}^k$,

$$(\mathcal{F}G(r,-\xi_1,\eta_2)-\mathcal{F}G(r,-\xi_1,\xi_2))^2 \le \frac{4\ C_1(T)}{1+|\xi_1|^2},$$

so by Assumption A_0 and the dominated convergence theorem, (6.17) converges to 0 as $\eta_2\to\xi_2$.

Now, let $x_1, y_1 \in \mathbb{R}^k$ and $\xi_2 \in \mathbb{R}^{d-k}$. Using (6.16), we find that

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}((Y(t,y_1,\xi_2) - Y(t,x_1,\xi_2))^2) \\
&\quad = \sup_{t\in[0,T]} 2\int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\, \mathcal{F}G(t-s,-\xi_1,\xi_2)^2\, (1-\cos(\xi_1\cdot(y_1-x_1))) \\
&\quad = 2\int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^T dr\, \mathcal{F}G(r,-\xi_1,\xi_2)^2\, (1-\cos(\xi_1\cdot(y_1-x_1))), \qquad (6.18)
\end{aligned}$$

where we have used the change of variable $r = t - s$ and the fact that $1 - \cos(\xi_1 \cdot (y_1 - x_1))$ is non-negative to remove the supremum in t. By continuity of the cosine function, the integrand in (6.18) converges to 0 as $y_1 \to x_1$, and by Lemma 5.4.1,

$$\mathcal{F}G(r,-\xi_1,\xi_2)^2\, (1-\cos(\xi_1\cdot(y_1-x_1)) \le \frac{2\, C_1(T)}{1+|\xi_1|^2}.$$

Therefore, Assumption A_0 and the dominated convergence theorem imply, as before, that expression (6.18) converges to 0 as $y_1 \to x_1$.

Finally, let $t, h \in \mathbb{R}_+$, $x_1 \in \mathbb{R}^k$ and $\xi_2 \in \mathbb{R}^{d-k}$. By (6.16),

$$\begin{aligned}
&\mathbb{E}((Y(t+h,x_1,\xi_2) - Y(t,x_1,\xi_2))^2) \\
&\quad = \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\, (\mathcal{F}G(t+h-s,-\xi_1,\xi_2) - \mathcal{F}G(t-s,-\xi_1,\xi_2))^2 \\
&\qquad + \int_{\mathbb{R}^k} \mu(d\xi_1) \int_t^{t+h} ds\, \mathcal{F}G(t+h-s,-\xi_1,\xi_2)^2 \\
&\quad = \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t dr\, (\mathcal{F}G(r+h,-\xi_1,\xi_2) - \mathcal{F}G(r,-\xi_1,\xi_2))^2 \qquad (6.19) \\
&\qquad + \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^h dq\, \mathcal{F}G(q,-\xi_1,\xi_2)^2. \qquad (6.20)
\end{aligned}$$

In the above, we have used the changes of variable $r = t - s$ and $q = t + h - s$. Since

$$(\mathcal{F}G(r+h,-\xi_1,\xi_2) - \mathcal{F}G(r,-\xi_1,\xi_2))^2 \underset{h\to 0}{\to} 0$$

and, by Lemma 5.4.1 and as in (6.15),

$$(\mathcal{F}G(r+h,-\xi_1,\xi_2) - \mathcal{F}G(r,-\xi_1,\xi_2))^2 \le \frac{4\, C_1(T+h_0)}{1+|\xi_1|^2}$$

for all $h \le h_0$, we obtain that the term in (6.19) converges to 0 as $h \to 0$ by Assumption A_0 and the dominated convergence theorem. Moreover, by Lemma 5.4.1, for all $h \le h_0$,

$$\begin{aligned}
\int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^h dq\, \mathcal{F}G(q,-\xi_1,\xi_2)^2 &\le \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^h dq\, \frac{C_1(h_0)}{1+|\xi_1|^2+|\xi_2|^2} \\
&\le h\, C_1(h_0) \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2},
\end{aligned}$$

so by Assumption A_0, (6.20) also converges to 0 as $h \to 0$, and this establishes the right-continuity in t of the process Y (in $L^2(\Omega)$).

To show the left-continuity, let us compute

$$\begin{aligned}
&\mathbb{E}((Y(t-h,x_1,\xi_2) - Y(t,x_1,\xi_2))^2) \\
&= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^{t-h} ds\ (\mathcal{F}G(t-h-s,-\xi_1,\xi_2) - \mathcal{F}G(t-s,-\xi_1,\xi_2))^2 \\
&\quad + \int_{\mathbb{R}^k} \mu(d\xi_1) \int_{t-h}^{t} ds\ \mathcal{F}G(t-s,-\xi_1,\xi_2)^2 \\
&= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^{t-h} dr\ (\mathcal{F}G(r,-\xi_1,\xi_2) - \mathcal{F}G(r+h,-\xi_1,\xi_2))^2 \\
&\quad + \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^{h} dq\ \mathcal{F}G(q,-\xi_1,\xi_2)^2.
\end{aligned}$$

In the above, we have again used the changes of variable $r = t-h-s$ and $q = t-s$. But this last expression is less than or equal to

$$\int_{\mathbb{R}^k} \mu(d\xi_1) \left(\int_0^t dr\ (\mathcal{F}G(r+h,-\xi_1,\xi_2) - \mathcal{F}G(r,-\xi_1,\xi_2))^2 + \int_0^h dq\ \mathcal{F}G(q,-\xi_1,\xi_2)^2 \right),$$

which converges to 0 as $h \to 0$ by same arguments as above. This completes the proof. □

Lemma 6.3.5. *Let us make Assumption A_0, let Y be the process defined in Lemma 6.3.3 and let u be the solution of equation (6.5). Then for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\langle u(t), \mathcal{F}_2\varphi \rangle = \int_{\mathbb{R}^k} dx_1 \int_{\mathbb{R}^{d-k}} d\xi_2\ Y(t,x_1,\xi_2)\ \varphi(x_1,\xi_2), \qquad \mathbb{P}-a.s.$$

Proof. By Lemma 6.3.3 and Remark 6.3.4, the integral on the right-hand side of the above equation is well defined. We show that both sides of the above equation are equal $\mathbb{P}$-a.s. by computing their variances and covariance, as in the proof of Theorem 5.3.1. By (6.7) and (6.13), we obtain that

$$\begin{aligned}
\mathbb{E}(|\langle u(t), \mathcal{F}_2\varphi\rangle|^2) &= \mathbb{E}\left(\left| \int_{[0,t]\times\mathbb{R}^k} M(ds,dx_1)\ (G(t-s) * \mathcal{F}_2\varphi)(x_1,0) \right|^2 \right) \\
&= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ |\mathcal{F}_1(G(t-s) * \mathcal{F}_2\varphi))(\xi_1,0)|^2
\end{aligned}$$

Since $\mathcal{F}_1 = \mathcal{F}_2^{-1}\mathcal{F}$ and $\mathcal{F}(G * H) = \mathcal{F}G \cdot \mathcal{F}H$, we can write that

$$\mathcal{F}_1(G(t-s) * \mathcal{F}_2\varphi)(\xi_1,0) = \mathcal{F}_2^{-1}(\mathcal{F}G(t-s) \cdot \mathcal{F}\mathcal{F}_2\varphi)(\xi_1,0).$$

Now, since by (4.1),

$$\mathcal{F}_2^{-1}\psi(\xi_1,0) = \frac{1}{(2\pi)^{d-k}} \int_{\mathbb{R}^{d-k}} d\xi_2\ \psi(\xi_1,\xi_2)$$

and

$$\mathcal{F}\mathcal{F}_2\varphi(\xi_1,\xi_2) = \mathcal{F}_1\mathcal{F}_2^2\varphi(\xi_1,\xi_2) = (2\pi)^{d-k}\ \mathcal{F}_1\varphi(\xi_1,-\xi_2),$$

we obtain that

$$\mathcal{F}_1(G(t-s)*\mathcal{F}_2\varphi)(\xi_1,0)=\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathcal{F}G(t-s,\xi_1,\xi_2)\ \mathcal{F}_1\varphi(\xi_1,-\xi_2), \tag{6.21}$$

so

$$\mathbb{E}(|\langle u(t),\mathcal{F}_2\varphi\rangle|^2)=\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t ds\left|\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathcal{F}G(t-s,\xi_1,\xi_2)\ \mathcal{F}_1\varphi(\xi_1,-\xi_2)\right|^2. \tag{6.22}$$

On the other hand, by Fubini's theorem and (6.16),

$$\begin{aligned}
&\mathbb{E}\left(\left|\int_{\mathbb{R}^k} dx_1\int_{\mathbb{R}^{d-k}} d\xi_2\ Y(t,x_1,\xi_2)\ \varphi(x_1,\xi_2)\right|^2\right)\\
&=\int_{\mathbb{R}^k} dx_1\int_{\mathbb{R}^{d-k}} d\xi_2\int_{\mathbb{R}^k} dy_1\int_{\mathbb{R}^{d-k}} d\eta_2\ \mathbb{E}(Y(t,x_1,\xi_2)\ Y(t,y_1,\eta_2))\ \varphi(x_1,\xi_2)\ \overline{\varphi(y_1,\eta_2)}\\
&=\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t ds\left|\int_{\mathbb{R}^k} dx_1\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathcal{F}G(t-s,-\xi_1,\xi_2)\ \chi_{x_1}(\xi_1)\ \varphi(x_1,\xi_2)\right|^2\\
&=\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t ds\left|\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathcal{F}G(t-s,-\xi_1,\xi_2)\ \mathcal{F}_1\varphi(\xi_1,\xi_2)\right|^2.
\end{aligned} \tag{6.23}$$

Using the change of variables $\xi_2\to-\xi_2$, we see that (6.22) and (6.23) are equal, since $\mathcal{F}G$ is symmetric in ξ.

Let us now compute, using Fubini's theorem and (6.13),

$$\begin{aligned}
&\mathbb{E}\left(\langle u(t),\mathcal{F}_2\varphi\rangle\cdot\overline{\int_{\mathbb{R}^k} dx_1\int_{\mathbb{R}^{d-k}} d\xi_2\ Y(t,x_1,\xi_2)\ \varphi(x_1,\xi_2)}\right)\\
&=\int_{\mathbb{R}^k} dx_1\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathbb{E}\bigg(\int_{[0,t]\times\mathbb{R}^k} M(ds,dy_1)\ (G(t-s)*\mathcal{F}_2\varphi)(y_1,0)\\
&\qquad\cdot\overline{\int_{[0,t]\times\mathbb{R}^k} M(ds,dy_1)\ \mathcal{F}_2G(t-s,x_1-y_1,\xi_2)}\bigg)\ \overline{\varphi(x_1,\xi_2)}\\
&=\int_{\mathbb{R}^k} dx_1\int_{\mathbb{R}^{d-k}} d\xi_2\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t ds\ \mathcal{F}_1(G(t-s)*\mathcal{F}_2\varphi)(\xi_1,0)\\
&\qquad\cdot\overline{\mathcal{F}G(t-s,-\xi_1,\xi_2)\ \chi_{x_1}(\xi_1)\ \varphi(x_1,\xi_2)}.
\end{aligned}$$

By (6.21) and Fubini's theorem, this last expression is equal to

$$\int_{\mathbb{R}^k}\mu(d\xi_1)\int_0^t ds\left(\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathcal{F}G(t-s,\xi_1,\xi_2)\ \mathcal{F}_1\varphi(\xi_1,-\xi_2)\cdot\overline{\int_{\mathbb{R}^{d-k}} d\xi_2\ \mathcal{F}G(t-s,-\xi_1,\xi_2)\ \mathcal{F}_1\varphi(\xi_1,\xi_2)}\right),$$

which is also equal to (6.22) and (6.23). This completes the proof. □

With these three lemmas in hand, we now prove the following proposition, which tells us that Assumption A_0 is a necessary and sufficient condition for the existence of the process Y.

Proposition 6.3.6. *Let u be the solution of equation (6.5). There exists a square integrable real-valued process $Y = \{ Y(t,x_1,\xi_2),\ (t,x_1,\xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\}$ such that the map $(t,x_1,\xi_2) \mapsto Y(t,x_1,\xi_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}$ to $L^2(\Omega)$ and*

$$\langle u(t), \mathcal{F}_2\varphi\rangle = \int_{\mathbb{R}^k} dx_1 \int_{\mathbb{R}^{d-k}} d\xi_2\, Y(t,x_1,\xi_2)\, \varphi(x_1,\xi_2), \qquad \mathbb{P}-a.s.,$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ if and only if Assumption A_0 is satisfied. Moreover, when Y exists, it is a centered Gaussian process whose covariance is given by formula (6.16).

Proof. The sufficiency of Assumption A_0 is a direct consequence of the three preceding lemmas. To show the necessity, fix $(t,x_1,\xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}$ and let $\varphi^{(n)}_{x_1,\xi_2} = \delta_{(x_1,\xi_2)} * \psi_n$, where (ψ_n) is a sequence of non-negative and compactly supported approximations of δ_0 in $\mathbb{R}^d$, so $\varphi^{(n)}_{x_1,\xi_2} \in \mathcal{S}(\mathbb{R}^d)$ for each n. The assumptions made on Y and Fubini's theorem imply that

$$\begin{aligned}
\mathbb{E}(|\langle u(t), \mathcal{F}_2\varphi^{(n)}_{x_1,\xi_2}\rangle|^2) &= \mathbb{E}\left(\left|\int_{\mathbb{R}^k} dy_1 \int_{\mathbb{R}^{d-k}} d\eta_2\, Y(t,y_1,\eta_2)\, \varphi^{(n)}_{x_1,\xi_2}(y_1,\eta_2)\right|^2\right)\\
&= \int_{\mathbb{R}^k} dy_1 \int_{\mathbb{R}^{d-k}} d\eta_2 \int_{\mathbb{R}^k} dz_1 \int_{\mathbb{R}^{d-k}} d\zeta_2\, \mathbb{E}(Y(t,y_1,\eta_2)\, Y(t,z_1,\zeta_2))\, \varphi^{(n)}_{x_1,\xi_2}(y_1,\eta_2)\, \overline{\varphi^{(n)}_{x_1,\xi_2}(z_1,\zeta_2)}\\
&\underset{n\to\infty}{\to} \mathbb{E}(Y(t,x_1,\xi_2)^2) < \infty. \qquad (6.24)
\end{aligned}$$

On the other hand, replacing φ by $\varphi^{(n)}_{x_1,\xi_2}$ in (6.22) gives

$$\mathbb{E}(|\langle u(t), \mathcal{F}_2\varphi\rangle|^2) = \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds \left|\int_{\mathbb{R}^{d-k}} d\eta_2\, \mathcal{F}G(t-s,\xi_1,\eta_2)\, \mathcal{F}_1\varphi^{(n)}_{x_1,\xi_2}(\xi_1,-\eta_2)\right|^2.$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^{d-k}} d\eta_2\, \mathcal{F}G(t-s,\xi_1,\eta_2)\, \mathcal{F}_1\varphi^{(n)}_{x_1,\xi_2}(\xi_1,-\eta_2)\\
&\quad= \int_{\mathbb{R}^k} dy_1 \int_{\mathbb{R}^{d-k}} d\eta_2\, \mathcal{F}G(t-s,\xi_1,\eta_2)\, \chi_{\xi_1}(y_1)\, \varphi^{(n)}_{x_1,\xi_2}(y_1,-\eta_2)\\
&\quad\underset{n\to\infty}{\to} \mathcal{F}G(t-s,\xi_1,-\xi_2)\, \chi_{\xi_1}(x_1),
\end{aligned}$$

for all $(s,\xi_1) \in [0,t] \times \mathbb{R}^k$, Fatou's lemma and Lemma 5.4.3 imply that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \mathcal{F}_2\varphi^{(n)}_{x_1,\xi_2}\rangle|^2) &\geq \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\, \mathcal{F}G(t-s,\xi_1,-\xi_2)^2\\
&\geq C_3(t) \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2+|\xi_2|^2}.
\end{aligned}$$

Since the above limit exists and is finite for all $\xi_2 \in \mathbb{R}^{d-k}$ by (6.24), it holds in particular for $\xi_2 = 0$, so Assumption A_0 is satisfied and this completes the proof. □

6.4 Regularity of the solution in the coordinates perpendicular to the k-plane

By the comment made at the beginning of Section 6.3, Assumption A_0 is the minimal condition for the solution of equation (6.5) to be a process U with values in some Sobolev space $H^\beta(\mathbb{R}^{d-k})$

in the coordinate x_2. We study thereafter more precisely this regularity, which is linked to the integrability in x_2 of the square of the process Y, namely the Fourier transform of the solution in x_2.

To this end, let us first define, for $\beta < 1 - \frac{d-k}{2}$ and $z \geq 1$,

$$\Lambda_\beta(z) = \begin{cases} \dfrac{1}{z^2}, & \text{if} \quad \beta < -\frac{d-k}{2}, \\[2ex] \dfrac{\ln(z^2)}{z^2}, & \text{if} \quad \beta = -\frac{d-k}{2}, \\[2ex] \dfrac{1}{z^{2(1-\frac{d-k}{2}-\beta)}}, & \text{if} \quad \beta \in \,]-\frac{d-k}{2}, 1-\frac{d-k}{2}[. \end{cases}$$

Let us then fix $\beta < 1 - \frac{d-k}{2}$ and make the following assumption on μ.

Assumption B_β.

$$\int_{\mathbb{R}^k} \mu(d\xi_1)\ \Lambda_\beta\left(\sqrt{1+|\xi_1|^2}\right) < \infty.$$

For clarity, we can rewrite this assumption in the separate three cases considered in the definition of the function Λ_β.

1) When $\beta < -\frac{d-k}{2}$, Assumption B_β is equivalent to **Assumption** A_0, namely

$$\int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2} < \infty.$$

2) When $\beta = -\frac{d-k}{2}$, Assumption B_β is equivalent to what we will term **Assumption** A_0':

$$\int_{\mathbb{R}^k} \frac{\mu(d\xi_1)\ \ln\left(1+|\xi_1|^2\right)}{1+|\xi_1|^2} < \infty.$$

3) When $\beta \in \,]-\frac{d-k}{2}, 1-\frac{d-k}{2}[$, Assumption B_β is equivalent to what we will term **Assumption** A_α, where $\alpha = \beta + \frac{d-k}{2} \in \,]0,1[$:

$$\int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{1-\alpha}} < \infty.$$

Note that, as for A_0, these assumptions can be reformulated into conditions on the covariance Γ when the latter is non-negative (see Section 6.6). One can already notice that when α tends to 1, Assumption A_α looks more and more like "μ is a finite measure", which says, by the classical Bochner theorem, that the measure Γ admits a density which is a uniformly continuous and bounded function.

Before stating the main result, we establish a technical lemma; for $\beta < 1 - \frac{d-k}{2}$ and $z \geq 1$, let us define

$$\begin{aligned} L_\beta(z) &= \int_{\mathbb{R}^{d-k}} d\xi_2 \ (1+|\xi_2|^2)^\beta \ \frac{1}{z^2+|\xi_2|^2} \\ &= \omega_{d-k-1} \int_0^\infty dr \ r^{d-k-1} \ (1+r^2)^\beta \ \frac{1}{z^2+r^2}, \end{aligned}$$

where ω_n denotes the area of the unit sphere S^n. Note that $L_\beta(\cdot)$ is a well defined and continuous function since $\beta < 1 - \frac{d-k}{2}$. The following lemma tells us that $L_\beta(z)$ behaves like $\Lambda_\beta(z)$ as $z \to \infty$.

Lemma 6.4.1. *Let $\beta < 1 - \frac{d-k}{2}$. There exists $R \geq 1$ and $0 < K_1 < K_2 < \infty$ such that*

$$K_1 \ \Lambda_\beta(z) \leq L_\beta(z) \leq K_2 \ \Lambda_\beta(z) \qquad \forall z \geq R.$$

Proof. We consider separately the three cases: $\beta < -\frac{d-k}{2}$, $\beta = -\frac{d-k}{2}$ and $\beta \in \left]-\frac{d-k}{2}, 1-\frac{d-k}{2}\right[$.

1) $\beta < -\frac{d-k}{2}$:

$$\frac{L_\beta(z)}{\Lambda_\beta(z)} = \omega_{d-k-1} \int_0^\infty dr \ f_\beta(r,z), \quad \text{where} \quad f_\beta(r,z) = r^{d-k-1} \ (1+r^2)^\beta \ \frac{z^2}{z^2+r^2}.$$

Since $\lim_{z\to\infty} f_\beta(r,z) = r^{d-k-1} \ (1+r^2)^\beta$, $|f_\beta(r,z)| \leq r^{d-k-1} \ (1+r^2)^\beta$ and

$$\int_0^\infty dr \ r^{d-k-1} \ (1+r^2)^\beta \leq \int_0^1 dr \ r^{d-k-1} + \int_1^\infty dr \ (2r)^{2\beta+d-k-1} < \infty,$$

(because $2\beta + d - k - 1 < -1$), we conclude by the dominated convergence theorem that

$$\lim_{z\to\infty} \frac{L_\beta(z)}{\Lambda_\beta(z)} = \omega_{d-k-1} \int_0^\infty dr \ r^{d-k-1} \ (1+r^2)^\beta = K \in (0,\infty),$$

and this implies the result.

2) $\beta = -\frac{d-k}{2}$:

$$\frac{L_\beta(z)}{\Lambda_\beta(z)} = \omega_{d-k-1} \int_0^\infty dr \ g_\beta(r,z), \quad \text{where} \quad g_\beta(r,z) = r^{d-k-1} \ (1+r^2)^\beta \ \frac{z^2}{z^2+r^2} \ \frac{1}{\ln(z^2)}.$$

Here we have

$$0 \leq \int_0^1 dr \ g_\beta(r,z) \leq \int_0^1 dr \ r^{d-k-1} \ \frac{1}{\ln(z^2)} \underset{z\to\infty}{\to} 0.$$

Let now

$$\tilde{K}(z) = \int_1^\infty \frac{dr}{r} \ \frac{z^2}{z^2+r^2} \ \frac{1}{\ln(z^2)} = \frac{\ln\left(1+z^2\right)}{2\ln(z^2)}$$

(for this last calculation, write $\frac{1}{r} \ \frac{z^2}{z^2+r^2} = \frac{1}{r} - \frac{r}{r^2+z^2}$ and integrate). Then

$$2^\beta \ \tilde{K}(z) \leq \int_1^\infty dr \ g_\beta(r,z) \leq \tilde{K}(z).$$

So the result follows, since

$$\ln(z^2) \leq \ln\left(1+z^2\right) \leq 2\ln(z^2), \qquad \forall z \geq \sqrt{2}.$$

3) $\beta \in]-\frac{d-k}{2}, 1-\frac{d-k}{2}[$:

$$\frac{L_\beta(z)}{\Lambda_\beta(z)} = \omega_{d-k-1} \int_0^\infty dr\ r^{d-k-1}\ (1+r^2)^\beta\ \frac{z^{2-(d-k)-2\beta}}{z^2+r^2} = \omega_{d-k-1} \int_0^\infty du\ h_\beta(u,z),$$

where we have used the change of variable $r = zu$ and

$$h_\beta(u,z) = u^{d-k-1}\ (\frac{1}{z^2}+u^2)^\beta\ \frac{1}{1+u^2}.$$

We see that $\lim\limits_{z\to\infty} h_\beta(u,z) = \dfrac{u^{d-k-1+2\beta}}{1+u^2}$. In order to apply the dominated convergence theorem, we need to consider two sub-cases:

3a) $\beta < 0$: $|h_\beta(u,z)| \leq \dfrac{u^{d-k-1+2\beta}}{1+u^2}$ and

$$\int_0^\infty du\ \frac{u^{d-k-1+2\beta}}{1+u^2} \leq \int_0^1 du\ u^{d-k-1+2\beta} + \int_1^\infty du\ u^{d-k-3+2\beta} < \infty,$$

since $d-k-1+2\beta > -1$ and $d-k-3+2\beta < -1$, by the assumptions made on β.

3b) $\beta \geq 0$: $|h_\beta(u,z)| \leq u^{d-k-1}\ (1+u^2)^{\beta-1}$ and

$$\int_0^\infty du\ u^{d-k-1}\ (1+u^2)^{\beta-1} \leq \int_0^1 du\ u^{d-k-1} + \int_1^\infty du\ (2u)^{d-k-3+2\beta} < \infty.$$

Thanks to these etimates, we can apply the dominated convergence theorem in both cases to conclude that

$$\lim_{z\to\infty} \frac{L_\beta(z)}{\Lambda_\beta(z)} = \omega_{d-k-1} \int_0^\infty du\ \frac{u^{d-k-1+2\beta}}{1+u^2} = \hat{K} \in (0,\infty),$$

and this completes the proof. □

We are now able to study the link between the regularity of the noise and the integrability in x_2 of the square of the process Y defined in Proposition 6.3.6, which in turn will give us an indication on the regularity of the solution in the same coordinate x_2.

Lemma 6.4.2. *Let $\beta < 1-\frac{d-k}{2}$. Under Assumption B_β, the process Y defined in Proposition 6.3.6 satisfies*

$$\mathbb{E}\left(\int_{\mathbb{R}^{d-k}} d\xi_2\ (1+|\xi_2|^2)^\beta\ Y(t,x_1,\xi_2)^2\right) < \infty, \qquad \forall (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k.$$

Proof. First note that if Assumption B_β is satisfied for $\beta < 1 - \frac{d-k}{2}$, then Assumption A_0 is, so the process Y is well defined by Proposition 6.3.6. Using Fubini's theorem and (6.16) gives

$$\begin{aligned}
&\mathbb{E}\left(\int_{\mathbb{R}^{d-k}} d\xi_2 \; (1+|\xi_2|^2)^\beta \; Y(t,x_1,\xi_2)^2\right) \\
&= \int_{\mathbb{R}^{d-k}} d\xi_2 \; (1+|\xi_2|^2)^\beta \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds \; \mathcal{F}G(t-s,-\xi_1,\xi_2)^2 \\
&\le C_2(t) \int_{\mathbb{R}^k} \mu(d\xi_1) \int_{\mathbb{R}^{d-k}} d\xi_2 \; \frac{(1+|\xi_2|^2)^\beta}{1+|\xi_1|^2+|\xi_2|^2} \\
&= C_2(t) \int_{\mathbb{R}^k} \mu(d\xi_1) \; L_\beta\left(\sqrt{1+|\xi_1|^2}\right),
\end{aligned}$$

by Lemma 5.4.2 and the definition of L_β. We now use Lemma 6.4.1 and Assumption B_β to conclude that

$$\begin{aligned}
&\mathbb{E}\left(\int_{\mathbb{R}^{d-k}} d\xi_2 \; (1+|\xi_2|^2)^\beta \; Y(t,x_1,\xi_2)^2\right) \\
&\le C_2(t)\left(\mu(B_1(0,R)) \; L_\beta(1) + K_2 \int_{B_1(0,R)^c} \mu(d\xi_1) \; \Lambda_\beta\left(\sqrt{1+|\xi_1|^2}\right)\right) < \infty,
\end{aligned}$$

which completes the proof. □

This allows us to establish the following result, which states the sufficiency and the necessity of Assumption B_β for the existence of a process with values in $H^\beta(\mathbb{R}^{d-k})$ in the coordinate x_2, which is the solution of equation (6.5).

Theorem 6.4.3. *Let u be the solution of equation (6.5) and let $\beta < 1 - \frac{d-k}{2}$. There exists a square integrable process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ with values in $H^\beta(\mathbb{R}^{d-k})$ such that the map $(t,x_1,\xi_2) \mapsto \mathcal{F}_2 U(t,x_1,\xi_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}$ to $L^2(\Omega)$ and*

$$\langle u(t),\varphi\rangle = \int_{\mathbb{R}^k} dx_1 \int_{\mathbb{R}^{d-k}} d\xi_2 \; \mathcal{F}_2 U(t,x_1,\xi_2) \; \mathcal{F}_2^{-1}\varphi(x_1,\xi_2), \qquad \mathbb{P}-a.s.,\ \forall t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d),$$

if and only if Assumption B_β is satisfied.

Remark 6.4.4. When $\beta \ge 0$, $U(t,x_1,\cdot) \in L^2(\mathbb{R})$, so the above equality can be rewritten in a more natural way:

$$\langle u(t),\varphi\rangle = \int_{\mathbb{R}^{d-1}} dx_1 \int_{\mathbb{R}} dx_2 \; U(t,x_1,x_2) \; \varphi(x_1,x_2).$$

Note that we still do not know if $U(t,x_1,x_2)$ is a well defined random variable for *every* $x_2 \in \mathbb{R}$, since we only know that $U(t,x_1,\cdot)$ takes its values in $L^2(\mathbb{R})$. This point will be clarified in the next chapter where we will see that the only problematic point is the point $x_2 = 0$.

Proof of Theorem 6.4.3. Let us first prove the sufficiency of Assumption B_β; since it implies Assumption A_0, let Y be the process whose existence is affirmed by Proposition 6.3.6. Define the process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ by

$$U(t,x_1,\cdot) = \mathcal{F}_2^{-1} Y(t,x_1,\cdot), \qquad (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k.$$

By Lemma 6.4.2, this is a square integrable process which takes its values in $H^\beta(\mathbb{R}^{d-k})$ and satisfies all the desired properties, by Proposition 6.3.6 and the fact that $\mathcal{F}_2 U(t,x_1,\xi_2) = Y(t,x_1,\xi_2)$.

In order to prove now the necessity of Assumption B_β, note that by Proposition 6.3.6, the existence of a process $\mathcal{F}_2 U$ which satisfies the above properties implies that Assumption A_0 is satisfied, so the process Y defined in the same theorem is well defined and the above assumptions tell us that

$$Y(t,x_1,\xi_2) = \mathcal{F}_2 U(t,x_1,\xi_2), \qquad \mathbb{P}-a.s., \ \forall (t,x_1,\xi_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k},$$

and that

$$\mathbb{E}\left(\int_{\mathbb{R}^{d-k}} d\xi_2 \ (1+|\xi_2|^2)^\beta \ Y(t,x_1,\xi_2)^2 \right) = \mathbb{E}\left(\|U(t,x_1)\|_\beta^2 \right) < \infty.$$

But on the other hand, by (6.16), Lemmas 5.4.3 and 6.4.1,

$$\begin{aligned} &\mathbb{E}\left(\int_{\mathbb{R}^{d-k}} d\xi_2 \ (1+|\xi_2|^2)^\beta \ Y(t,x_1,\xi_2)^2 \right) \\ &= \int_{\mathbb{R}^{d-k}} d\xi_2 \ (1+|\xi_2|^2)^\beta \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds \ \mathcal{F}G(t-s,-\xi_1,\xi_2)^2 \\ &\geq \int_{\mathbb{R}^{d-k}} d\xi_2 \ (1+|\xi_2|^2)^\beta \ C_3(t) \int_{\mathbb{R}^k} \frac{\mu(d\xi_1)}{1+|\xi_1|^2+|\xi_2|^2} \\ &= C_3(t) \int_{\mathbb{R}^k} \mu(d\xi_1) \ L_\beta\left(\sqrt{1+|\xi_1|^2}\right) \\ &\geq C_3(t) \ K_1 \int_{B_1(0,R)^c} \mu(d\xi_1) \ \Lambda_\beta\left(\sqrt{1+|\xi_1|^2}\right), \end{aligned}$$

so Assumption B_β is satisfied and this completes the proof. □

Remark 6.4.5. We need here to make clear in what sense Assumption B_β is optimal. In the preceding theorem, we have shown that Assumption B_β is necessary under the assumption that there exists a real-valued process which is the Fourier transform of the solution in x_2. But this latter assumption is quite strong, since it implies that the solution has some L^2-type behavior in x_2 (that is, belongs to some $H^\beta(\mathbb{R}^{d-k})$), which is a priori not satisfied by any real-valued function. Nevertheless, the results of the next chapter will confirm that this assumption is not a restriction and that Assumption B_β is optimal in the case of the hyperbolic equation. On the contrary, Assumption B_β is not optimal in the case of the heat equation, since in this case, even for rough noises (like white noise) on a k-plane, there exists a real-valued process which is the solution of the equation and which has a strongly singular behavior near the k-plane $x_2 = 0$ (see Chapter 9).

6.5 Summary

Using the above rewriting of Assumption B_β into three separate cases, we can make Theorem 6.4.3 more explicit.

1) The solution of equation (6.5) is a process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ with values in $H^{-\frac{d-k}{2}-\varepsilon}(\mathbb{R}^{d-k})$, for some $\varepsilon > 0$, if and only if Assumption A_0 is satisfied.

Remark 6.5.1. As mentioned in Remark 6.3.1, Assumption A_0 is also the necessary and sufficient condition for the existence of a real-valued process which is the solution of a hyperbolic equation in $\mathbb{R}^k$ driven by spatially homogeneous noise with spectral measure μ. In the present case, the equation is the hyperbolic equation in $\mathbb{R}^d$ driven by the noise term $\dot{F}(t,x_1)\ \delta_0(x_2)$. Noting that $\delta_0 \in H^{-\frac{d-k}{2}-\varepsilon}(\mathbb{R}^{d-k})$, for any $\varepsilon > 0$, allows us to see the clear connection between these two reults.

2) The solution is a process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ with values in $H^{-\frac{d-k}{2}}(\mathbb{R}^{d-k})$ if and only if Assumption A_0' is satisfied.

3) For $\alpha \in\ ]0,1[$, the solution is a process $U = \{U(t,x_1),\ (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ with values in $H^{\alpha-\frac{d-k}{2}}(\mathbb{R}^{d-k})$ if and only if Assumption A_α is satisfied.

Remark 6.5.2. The exponent $\beta = \alpha - \frac{d-k}{2}$ of the Sobolev space in which the process U takes its values can be non-negative (that is, the solution can be a function-valued process) only when $k = d-1$ (that is, in the case of a noise concentrated on a hyperplane), in which case Assumption B_β becomes

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}} < \infty,$$

for $\beta \in [0, \frac{1}{2}[$. In the next chapter, we will see that this assumption implies another kind of regularity of the solution.

6.6 Reformulation of the conditions on the spectral measure

Let us now assume that Γ is a non-negative measure on $\mathbb{R}^k$. We will give here conditions on the covariance Γ which are (almost) equivalent to condition A_α for $\alpha \in [0,1[$, using the results of Section 4.4.

First note that Assumption A_α is condition (4.9) with d replaced by k and η replaced by $1-\alpha$. Using then (4.10) and Proposition 4.4.1, we obtain that Assumption A_α is equivalent to

$$\int_{\mathbb{R}^k} \Gamma(dx_1)\ G_{k,1-\alpha}(x_1) < \infty, \tag{6.25}$$

modulo the boundedness assumption of Proposition 4.4.1 for the case $\alpha \neq 0$. Following the argument of Section 4.4, let us now make this last condition more explicit.

- If $k = 1$ and $\alpha < \frac{1}{2}$, then (6.25) imposes no restriction on the covariance Γ.

- If $k = 1$ and $\alpha = \frac{1}{2}$, or $k = 2$ and $\alpha = 0$, then (6.25) is satisfied if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1)\ \ln\left(\frac{1}{|x_1|}\right) < \infty.$$

- Otherwise, (6.25) is satisfied if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1)\ \frac{1}{|x_1|^{2\alpha+k-2}} < \infty.$$

In the case where $\Gamma(dx_1) = f(|x_1|)\, dx_1$, with f a continuous function on $]0,\infty[$, this implies the following.

- If $k = 1$ and $\alpha < \frac{1}{2}$, then (6.25) imposes no restriction on f.

- If $k = 1$ and $\alpha = \frac{1}{2}$, then (6.25) is satisfied if and only if

$$\int_0^1 dr\ f(r)\ \ln\left(\frac{1}{r}\right) < \infty.$$

- If $k = 2$ and $\alpha = 0$, then (6.25) is satisfied if and only if

$$\int_0^1 dr\ f(r)\ r\ \ln\left(\frac{1}{r}\right) < \infty.$$

- Otherwise, (6.25) is satisfied if and only if

$$\int_0^1 dr\ f(r)\ \frac{1}{r^{2\alpha-1}} < \infty.$$

Finally, the reformulation of condition A_0', though not established because of technical difficulties, is conjectured to give the following.

- If $k = 1$, then Assumption A_0' imposes no restriction on the covariance Γ (this is clear because assumption A_0' implies Assumption A_0 and is implied by Assumption $A_{\frac{1}{4}}$, for example).

- If $k = 2$, we expect that Assumption A_0' is satisfied (perhaps modulo a boundedness assumption) if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1)\ \ln\left(\frac{1}{|x_1|}\right)^2 < \infty.$$

- If $k > 2$, we expect that Assumption A_0' is satisfied (perhaps modulo a boundedness assumption) if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1)\ \frac{1}{|x_1|^{k-2}}\ \ln\left(\frac{1}{|x_1|}\right) < \infty.$$

Chapter 7

Existence of a real-valued solution

A first remark concerns the expression: "real-valued solution". It is intended here to be opposed to "distribution-valued solution" (not to "complex-valued solution"). To be precise, a real-valued solution is a real-valued process X which represents the weak solution u of equation (6.2) in the sense that

$$\langle u(t), \varphi \rangle = \int_{\mathbb{R}^d} dx\; X(t,x)\; \varphi(x), \qquad \mathbb{P} - a.s.,\ \forall t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

In this chapter, we answer two questions: when is the weak solution of equation (6.2) a real-valued process? How regular then is this process? In the case of a noise on a hyperplane (that is, when $k = d-1$), we will see that the answer to the first question is positive when Assumption B_β of Section 6.4 is satisfied for $\beta \in [0, \frac{1}{2}[$ and that the regularity of the solution depends on β, whereas in the case of a noise on a lower dimensional plane (that is, when $k < d-1$), there does not exist a real-valued process which is the weak solution of equation (6.2).

We begin with the case of a noise on a hyperplane (that is, when $k = d-1$). The techniques that we use are similar to those of Section 6.3. However, we first need to establish some properties of $\mathcal{F}_1 G$, the Fourier transform in the first $d-1$ coordinates of x of the solution of equation (5.15), since this term will appear in the expectation of the square of the real-valued process which is the weak solution of equation (6.2).

For the same kind of technical reasons as in Chapter 5, we now restrict ourselves to the case where either $d \in \{2,3\}$ and a, b are any real numbers, or $d \in \{4,5\}$ and $a = b = 0$ (see Remark A.2.1).

7.1 Fourier transform of the Green kernel in x_1

Using (4.1), (5.18) and [45, formulas I.5.83 and I.7.61], we obtain the Fourier transform of G in the first $d-1$ coordinates of x: for $(t, \xi_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}$, we have

$$\mathcal{F}_1 G(t,\xi_1,x_2) = \mathcal{F}_2^{-1}(\mathcal{F}G(t,\xi_1,\cdot))(x_2) = \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2\ \mathcal{F}G(t,\xi_1,\xi_2)\ \chi_{-x_2}(\xi_2)$$
$$= \begin{cases} \dfrac{e^{-at}}{2}\ J_0\left(\sqrt{(|\xi_1|^2+b-a^2)\ (t^2-x_2^2)}\right)\ 1_{\{|x_2|\ <\ t\}}, & \text{if} \quad |\xi_1|^2 \geq a^2-b, \\ \dfrac{e^{-at}}{2}\ I_0\left(\sqrt{(a^2-b-|\xi_1|^2)\ (t^2-x_2^2)}\right)\ 1_{\{|x_2|\ <\ t\}}, & \text{if} \quad a^2-b>0 \text{ and } |\xi_1|^2 < a^2-b, \end{cases} \tag{7.1}$$

where J_0 and I_0 are the zero order regular and modified Bessel functions of the first kind (see Appendix B for an overview of their basic properties which will be used below).

$\mathcal{F}_1 G$ is a real-valued and Borel-measurable function, which is bounded on $[0,T]\times\mathbb{R}^{d-1}\times\mathbb{R}$ for all $T>0$, and it is symmetric and infinitely differentiable in ξ_1, since for all $t\in\mathbb{R}_+$ and $x_2\in\mathbb{R}$, $\mathcal{F}_1 G(t,\cdot,x_2)$ is an analytic function on $\mathbb{R}^{d-1}$, whose Taylor series is given by

$$\mathcal{F}_1 G(t,\xi_1,x_2) = \frac{1}{2}\sum_{n\in\mathbb{N}} (-1)^n\ \frac{(t^2-x_2^2)^n}{2^{2n}\ (n!)^2}\ (|\xi_1|^2+b-a^2)^n\ 1_{\{|x_2|\ <\ t\}}, \qquad \forall \xi_1\in\mathbb{R}^{d-1}.$$

Moreover, since $\mathcal{F}_1 G(t,\cdot,x_2)$ and all its derivatives in ξ_1 vanish at infinity, $\mathcal{F}_1 G(t,\cdot,x_2)\in O_M(\mathbb{R}^{d-1})$, so $G(t,\cdot,x_2)\in O_C'(\mathbb{R}^{d-1})$ by (4.2).

From the explicit expressions of G listed in Appendix A, we also deduce that for all $t\in\mathbb{R}_+$ and $x_2\in\mathbb{R}$, $G(t,\cdot,x_2)$ is a finite order distribution with compact support on $\mathbb{R}^{d-1}$; furthermore, for all $t\in\mathbb{R}_+$, there exist $K_1(t)>0$ and $N_1\in\mathbb{N}$ such that

$$\sup_{s\in[0,t],\ x_2\in\mathbb{R}} |G(s,\varphi,x_2)| \leq K_1(t)\sum_{|\underline{n}_1|\leq N_1}\ \sup_{x_1\in\overline{B_1(0,t)}} |\partial^{\underline{n}_1}\varphi(x_1)|, \qquad \forall\varphi\in\mathcal{S}(\mathbb{R}^{d-1}), \tag{7.2}$$

where $\underline{n}_1$ denotes a multi-index in $\mathbb{N}^{d-1}$.

$\mathcal{F}_1 G$ has also the following properties, which will be used in the next section.

Lemma 7.1.1. *For all $t>0$, there exists $C_4(t)>0$ such that*

$$\mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{C_4(t)}{\sqrt{1+|\xi_1|^2}}\ \frac{1}{\sqrt{s^2-x_2^2}}\ 1_{\{|x_2|\ <\ s\}}, \quad \forall s\in[0,t],\ \xi_1\in\mathbb{R}^{d-1},\ x_2\in\mathbb{R}.$$

Proof. If $|\xi_1|^2 \geq 2(a^2-b)+1$, then since $J_0(r)^2\leq \dfrac{C}{r}$ for all $r>0$ and (5.21) implies that

$$\sqrt{|\xi_1|^2+b-a^2} \geq \sqrt{\frac{1+|\xi_1|^2}{2}}, \tag{7.3}$$

we obtain that

$$\begin{aligned} \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \frac{e^{-2as}}{4}\ J_0^2\left(\sqrt{(|\xi_1|^2+b-a^2)\ (s^2-x_2^2)}\right)\ 1_{\{|x_2|\ <\ s\}} \\ &\leq \frac{e^{2a^- t}}{4}\ \frac{C\sqrt{2}}{\sqrt{1+|\xi_1|^2}}\ \frac{1}{\sqrt{s^2-x_2^2}}\ 1_{\{|x_2|\ <\ s\}}. \end{aligned}$$

If $2(a^2-b)+1 \geq 0$ and $a^2-b \leq |\xi_1|^2 \leq 2(a^2-b)+1$, then since $J_0(r)^2 \leq 1$ for all $r \geq 0$, we obtain

$$\begin{aligned}
\mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \frac{e^{-2as}}{4}\, J_0^2\left(\sqrt{(|\xi_1|^2+b-a^2)\,(s^2-x_2^2)}\right)\, 1_{\{|x_2|<s\}} \\
&\leq \frac{e^{2a^- t}}{4}\, 1_{\{|x_2|<s\}} \\
&\leq \frac{e^{2a^- t}}{4}\, \frac{t}{\sqrt{s^2-x_2^2}}\, 1_{\{|x_2|<s\}},
\end{aligned}$$

since $\sqrt{s^2-x_2^2} \leq s \leq t$. Finally, if $a^2-b \geq 0$ and $|\xi_1|^2 \leq a^2-b$, then since $I_0(r)^2 \leq C\, e^{2r}$ for all $r \geq 0$, we have

$$\begin{aligned}
\mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \frac{e^{-2as}}{4}\, I_0^2\left(\sqrt{(a^2-b-|\xi_1|^2)\,(s^2-x_2^2)}\right)\, 1_{\{|x_2|<s\}} \\
&\leq \frac{e^{2a^- t}}{4}\, C\, e^{2\sqrt{a^2-b}\,t}\, 1_{\{|x_2|<s\}} \\
&\leq \frac{e^{2a^- t}}{4}\, C\, e^{2\sqrt{a^2-b}\,t}\, \frac{t}{\sqrt{s^2-x_2^2}}\, 1_{\{|x_2|<s\}},
\end{aligned}$$

since $\sqrt{s^2-x_2^2} \leq s \leq t$ as before. The proof now ends as the proof of Lemma 5.4.1. □

The preceding lemma, as Lemma 5.4.1, will be used for rather technical purposes. A direct consequence is the following upper bound.

Lemma 7.1.2. *For all $t>0$, there exists $C_5(t)>0$ such that*

$$\int_0^t ds\, \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{C_5(t)}{\sqrt{1+|\xi_1|^2}}\, \operatorname{arccosh}\left(\frac{t}{|x_2|}\right)\, 1_{\{|x_2|<t\}}, \quad \forall \xi_1 \in \mathbb{R}^{d-1},\ x_2 \in \mathbb{R}^*.$$

Proof. We obtain this inequality by a simple integration in s of the result of Lemma 7.1.1. □

As Lemma 5.4.3, the following lemma gives a corresponding lower bound.

Lemma 7.1.3. *For all $t>0$ and $x_2 \in \mathbb{R}$ such that $0<|x_2|<t$, there exists $C_6(t,x_2)>0$ such that*

$$\int_0^t ds\, \mathcal{F}_1 G(s,\xi_1,x_2)^2 \geq \frac{C_6(t,x_2)}{\sqrt{1+|\xi_1|^2}}, \qquad \forall \xi_1 \in \mathbb{R}^{d-1}.$$

Proof. If $|\xi_1|^2 \geq a^2-b+\frac{1}{t^2-x_2^2}$ (recall that $0<|x_2|<t$ by assumption), then

$$\begin{aligned}
\int_0^t ds\, \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \int_0^t ds\, \frac{e^{-2as}}{4}\, J_0^2\left(\sqrt{(|\xi_1|^2+b-a^2)\,(s^2-x_2^2)}\right)\, 1_{\{|x_2|<s\}} \\
&\geq \frac{e^{-2a^+ t}}{4} \int_{|x_2|}^t ds\, J_0^2\left(\sqrt{(|\xi_1|^2+b-a^2)\,(s^2-x_2^2)}\right).
\end{aligned}$$

Use the change of variables $r=\sqrt{(|\xi_1|^2+b-a^2)\,(s^2-x_2^2)}$, so that

$$ds = \frac{r\,dr}{s\,(|\xi_1|^2+b-a^2)} \geq \frac{r\,dr}{t\,(|\xi_1|^2+b-a^2)},$$

and set $R = \sqrt{(|\xi_1|^2 + b - a^2)\ (t^2 - x_2^2)}$ to see that

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \geq \frac{e^{-2a^+t}}{4t}\ \frac{1}{|\xi_1|^2 + b - a^2} \int_0^R dr\ r\ J_0(r)^2.$$

Since $R \geq 1$ and using Lemma B.2.1, we obtain that

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \geq \frac{C\ e^{-2a^+t}}{4t}\ \frac{\sqrt{(|\xi_1|^2 + b - a^2)\ (t^2 - x_2^2)}}{|\xi_1|^2 + b - a^2}.$$

Moreover, by (5.22),

$$\sqrt{|\xi_1|^2 + b - a^2} \leq (1 \vee \sqrt{b - a^2})\ \sqrt{1 + |\xi_1|^2}, \tag{7.4}$$

so

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \geq \frac{C\ e^{-2a^+t}\ \sqrt{t^2 - x_2^2}}{4t}\ \frac{1}{1 \vee \sqrt{b - a^2}}\ \frac{1}{\sqrt{1 + |\xi_1|^2}}.$$

If $a^2 - b + \frac{1}{t^2 - x_2^2} \geq 0$ and $a^2 - b \leq |\xi_1|^2 \leq a^2 - b + \frac{1}{t^2 - x_2^2}$, then

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \int_{|x_2|}^t ds\ \frac{e^{-2as}}{4}\ J_0\left(\sqrt{(|\xi_1|^2 + b - a^2)\ (s^2 - x_2^2)}\right)^2 \\ &\geq e^{-2a^+t}\ (t - |x_2|)\ J_0(1)^2,\end{aligned}$$

since $\sqrt{(s^2 - x_2^2)\ (|\xi_1|^2 + b - a^2)} \leq 1$ for all $s \in [0,t]$ and $J_0(r)^2 \geq J_0(1)^2 > 0$ for all $r \in [0,1]$. Finally, if $a^2 - b \geq 0$ and $|\xi_1|^2 \leq a^2 - b$, then

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \int_{|x_2|}^t ds\ \frac{e^{-2as}}{4}\ I_0\left(\sqrt{(a^2 - b - |\xi_1|^2)\ (s^2 - x_2^2)}\right)^2 \\ &\geq e^{-2a^+t}\ (t - |x_2|)\end{aligned}$$

since $I_0(r)^2 \geq 1$ for all $r \geq 0$, and the proof ends as the proof of Lemma 5.4.3. □

7.2 Optimal condition on the spectral measure

Similarly to Section 6.3, we will see here that there exists a real-valued process defined outside the hyperplane $x_2 = 0$ which is the weak solution of equation (6.2) if and only if **Assumption** B_0 of the preceding chapter is satisfied, namely

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1 + |\xi_1|^2}} < \infty.$$

Note that because of the square root, this condition is stronger than the one obtained for the equation driven by spatially homogeneous noise (see [15, 30]), but this is quite normal since the noise considered here, being concentrated on a hyperplane, is by nature more singular than a spatially homogeneous one.

In Section 7.5, we give a reformulation of Assumption B_0 into a condition on the covariance Γ, when the latter is non-negative. One can already notice that the Lebesgue measure on $\mathbb{R}^{d-1}$

(which is the spectral measure of *white noise* on $\mathbb{R}^{d-1}$) does not satisfy this condition for any dimension d greater than 1. This result is then completely different from the one obtained for the heat equation, for which there always exists a real-valued solution outside the hyperplane $x_2 = 0$ (see Chapter 9).

We now prove the sufficiency of Assumption B_0 through the following three lemmas.

Lemma 7.2.1. *Under Assumption B_0 and for $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$, the function $\phi_{t,x_1,x_2} : [0,t] \to O'_C(\mathbb{R}^{d-1})$ defined by*

$$\phi_{t,x_1,x_2}(s,\cdot) = G(t-s, x_1 - \cdot, x_2), \qquad s \in [0,t],$$

belongs to H_t.

Proof. Theorem 6.2.2 does not apply here (condition (6.14) is not satisfied, mainly because $\mathcal{F}_1 G$ is not continuous in s), so we need to show directly that $\phi_{t,x_1,x_2} \in H_t$, using the definition of this space (see Section 6.2). Fix therefore $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$. Note that for all $s \in [0,t]$, $\xi_1 \in \mathbb{R}^{d-1}$,

$$\mathcal{F}_1 \phi_{t,x_1,x_2}(s,\xi_1) = \mathcal{F}_1 G(t-s, -\xi_1, x_2)\ \chi_{x_1}(\xi_1).$$

Thus, for all $s \in [0,t]$, $\mathcal{F}_1 \phi_{t,x_1,x_2}(s,\cdot) \in O_M(\mathbb{R}^{d-1})$ (see Section 7.1) and this implies that $\phi_{t,x_1,x_2}(s,\cdot) \in O'_C(\mathbb{R}^{d-1})$, by (4.2). Moreover, $\mathcal{F}_1 \phi_{t,x_1,x_2}$ is a Borel-measurable function and using Lemma 7.1.2 and Assumption B_0, we obtain that

$$\begin{aligned} \|\phi_{t,x_1,x_2}\|_t^2 &= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ \mathcal{F}_1 G(t-s, -\xi_1, x_2)^2 \\ &\le C_5(t) \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1+|\xi_1|^2}}\ \text{arccosh}\left(\frac{t}{|x_2|}\right)\ 1_{\{|x_2| < t\}} < \infty, \end{aligned} \tag{7.5}$$

since $x_2 \neq 0$ by the assumption made above. Let us now define

$$\phi^{(n)}_{t,x_1,x_2}(s, y_1) = (\phi_{t,x_1,x_2}(s) *_1 \psi_n)\ (y_1), \qquad s \in [0,t],\ y_1 \in \mathbb{R}^{d-1}, \tag{7.6}$$

where (ψ_n) is a sequence of non-negative and compactly supported approximations of δ_0 in $\mathbb{R}^{d-1}$, which satisfies

$$\int_{\mathbb{R}^{d-1}} dx_1\ \psi_n(x_1) = 1, \quad \text{so} \quad |\mathcal{F}_1 \psi_n(\xi_1)| \le 1, \quad \forall \xi_1 \in \mathbb{R}^{d-1}.$$

For each n, we have

$$\mathcal{F}_1 \phi^{(n)}_{t,x_1,x_2}(s,\xi_1) = \mathcal{F}_1 \phi_{t,x_1,x_2}(s,\xi_1)\ \mathcal{F}_1 \psi_n(\xi_1),$$

which implies that

$$\|\phi_{t,x_1,x_2} - \phi^{(n)}_{t,x_1,x_2}\|_t^2 = \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\ |\mathcal{F}_1 \phi_{t,x_1,x_2}(s,\xi_1)|^2\ |1 - \mathcal{F}_1 \psi_n(\xi_1)|^2.$$

Using the dominated convergence theorem together with the following facts:

$$\mathcal{F}_1 \psi_n(\xi_1) \underset{n\to\infty}{\to} 1, \quad |1 - \mathcal{F}_1 \psi_n(\xi_1)| \le 2 \quad \text{and} \quad \|\phi_{t,x_1,x_2}\|_t < \infty,$$

we conclude that

$$\|\phi_{t,x_1,x_2} - \phi^{(n)}_{t,x_1,x_2}\|_t \underset{n\to\infty}{\to} 0.$$

It remains to check that $\phi^{(n)}_{t,x_1,x_2} \in H_{t,0}$ for each n. By (7.6) and definition of $\phi^{(n)}_{t,x_1,x_2}$, $\phi^{(n)}_{t,x_1,x_2}$ is a Borel-measurable function and for all $s \in [0,t]$, $\phi^{(n)}_{t,x_1,x_2}(s,\cdot) \in \mathcal{S}(\mathbb{R}^{d-1})$, since $\phi_{t,x_1,x_2}(s,\cdot) \in O'_C(\mathbb{R}^{d-1})$. The last condition to verify is that $\|\phi^{(n)}_{t,x_1,x_2}\|_{t,+} < \infty$.

We also deduce from the definition of $\phi^{(n)}_{t,x_1,x_2}$ that for all $s \in [0,t]$, $\phi^{(n)}_{t,x_1,x_2}(s,\cdot)$ is compactly supported, and therefore belongs to $C_0^\infty(\mathbb{R}^{d-1})$ and so does $\phi^{(n)}_{t,x_1,x_2}(s,\cdot) *_1 \phi^{(n)}_{t,x_1,x_2}(s,\cdot)$. Moreover, by estimate (7.2), there exists $R_1 > 0$ such that

$$\sup_{s\in[0,t]} (|\phi^{(n)}_{t,x_1,x_2}(s,\cdot)| *_1 |\phi^{(n)}_{t,x_1,x_2}(s,\cdot)|)(z_1) = 0, \qquad \forall z_1 \in \mathbb{R}^{d-1} \text{ with } |z_1| > R_1.$$

This implies that

$$\|\phi^{(n)}_{t,x_1,x_2}\|^2_{t,+} = \int_0^t ds \int_{\mathbb{R}^{d-1}} \nu(dz_1)\ (|\phi^{(n)}_{t,x_1,x_2}(s,\cdot)| *_1 |\phi^{(n)}_{t,x_1,x_2}(s,\cdot))|(z_1) < \infty,$$

which completes the proof. □

Lemma 7.2.2. *Let M be the worthy martingale measure defined in Section 6.1 in the case $k = d-1$. Under Assumption B_0, the real-valued process $X = \{X(t,x_1,x_2),\ (t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*\}$ defined by*

$$X(t,x_1,x_2) = \int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dy_1)\ G(t-s,x_1-y_1,x_2), \qquad (t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*,$$

is a centered Gaussian process whose covariance is given by

$$\begin{aligned}&\mathbb{E}(X(t,x_1,x_2)\ X(s,y_1,y_2))\\ &= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^{t\wedge s} dr\ \mathcal{F}_1G(t-r,-\xi_1,x_2)\ \mathcal{F}_1G(s-r,-\xi_1,y_2)\ \chi_{x_1-y_1}(\xi_1),\end{aligned} \tag{7.7}$$

and such that the map $(t,x_1,x_2) \mapsto X(t,x_1,x_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ to $L^2(\Omega)$.

Remark 7.2.3. This result and [42, Prop. 3.6 and Cor. 3.8] imply that the process X admits a modification $\tilde{X}$ such that the map $(t,x_1,x_2,\omega) \mapsto \tilde{X}(t,x_1,x_2,\omega)$ is jointly measurable. We will implicitely consider this modification in the following.

Proof of Lemma 7.2.2. First note that the proof follows exactly the same scheme as the proof of Lemma 6.3.3, but the estimates are quite different.

By Lemma 7.2.1, the process X is well defined. The fact that X is a centered Gaussian process with the covariance given above follows easily from the isometry (6.13), and since μ and $\mathcal{F}_1G$ are symmetric in ξ_1, (7.7) is equal to

$$\int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^{t\wedge s} dr\ \mathcal{F}_1G(t-r,-\xi_1,x_2)\ \mathcal{F}_1G(s-r,-\xi_1,y_2)\ \cos(\xi_1\cdot(x_1-y_1)),$$

so X is real-valued.

In order to show that the map $(t, x_1, x_2) \mapsto X(t, x_1, x_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ to $L^2(\Omega)$, we show that for all $T > 0$, it is continuous from $[0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ to $L^2(\Omega)$, showing first that the map $x_2 \mapsto X(t, x_1, x_2)$ is continuous in $L^2(\Omega)$ uniformly in $(t, x_1) \in [0, T] \times \mathbb{R}^{d-1}$, then that for fixed $x_2 \in \mathbb{R}^*$, the map $x_1 \mapsto X(t, x_1, x_2)$ is continuous in $L^2(\Omega)$ uniformly in $t \in [0, T]$ and finally that for fixed $(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}^*$, the map $t \mapsto X(t, x_1, x_2)$ is continuous in $L^2(\Omega)$.

Therefore, let x_2, $y_2 \in \mathbb{R}^*$. Using (7.7) and arguments similar to those that led to (6.17), we obtain that

$$\begin{aligned}
&\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \mathbb{E}((X(t, x_1, y_2) - X(t, x_1, x_2))^2) \\
&\quad = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^T dr\ (\mathcal{F}_1 G(r, -\xi_1, y_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2. \qquad (7.8)
\end{aligned}$$

We will show in two steps that this expression converges to 0 as $y_2 \to x_2$. First note that for each $\xi_1 \in \mathbb{R}^{d-1}$ and $r \neq |x_2|$,

$$(\mathcal{F}_1 G(r, -\xi_1, y_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2 \underset{y_2 \to x_2}{\to} 0.$$

Moreover, since $\mathcal{F}_1 G$ is bounded on $[0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}$, we obtain from the dominated convergence theorem that

$$\int_0^T dr\ (\mathcal{F}_1 G(r, -\xi_1, y_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2 \underset{y_2 \to x_2}{\to} 0.$$

But for $\varepsilon \in]0, |x_2|[$ and $|y_2 - x_2| < \varepsilon$, we obtain by Lemma 7.1.2 that

$$\begin{aligned}
&\int_0^T dr\ (\mathcal{F}_1 G(r, -\xi_1, y_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2 \\
&\quad \leq \frac{2\, C_5(T)}{\sqrt{1 + |\xi_1|^2}} \left(\operatorname{arccosh}\left(\frac{T}{|y_2|}\right) 1_{\{|y_2| < T\}} + \operatorname{arccosh}\left(\frac{T}{|x_2|}\right) 1_{\{|x_2| < T\}} \right) \\
&\quad \leq \frac{2\, C_5(T)}{\sqrt{1 + |\xi_1|^2}} \left(\operatorname{arccosh}\left(\frac{T}{|x_2| - \varepsilon}\right) 1_{\{|x_2| - \varepsilon < T\}} + \operatorname{arccosh}\left(\frac{T}{|x_2|}\right) 1_{\{|x_2| < T\}} \right),
\end{aligned}$$

since $|y_2| > |x_2| - \epsilon$. So by Assumption B_0 and the dominated convergence theorem, (7.8) converges to 0 as $y_2 \to x_2$.

Now, let x_1, $y_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}^*$. As in the proof of (6.18), (7.7) leads to

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}((X(t, y_1, x_2) - X(t, x_1, x_2))^2) \\
&\quad = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^T dr\ \mathcal{F}_1 G(r, -\xi_1, x_2)^2\ 2\,(1 - \cos(\xi_1 \cdot (y_1 - x_1))). \qquad (7.9)
\end{aligned}$$

By continuity of the cosine function, the integrand in (7.9) converges to 0 as $y_1 \to x_1$ and by Lemma 7.1.1,

$$\mathcal{F}_1 G(r, -\xi_1, x_2)^2 \; 2\,(1 - \cos(\xi_1 \cdot (y_1 - x_1)) \le \frac{4\, C_4(T)}{\sqrt{1+|\xi_1|^2}} \; \frac{1}{\sqrt{r^2 - x_2^2}} \; 1_{\{|x_2| < r\}},$$

so using again Assumption B_0, the fact that, since $x_2 \neq 0$,

$$\int_0^T dr \; \frac{1}{\sqrt{r^2 - x_2^2}} \; 1_{\{|x_2| < r\}} = \operatorname{arccosh}\left(\frac{T}{|x_2|}\right) \; 1_{\{|x_2| < T\}} < \infty,$$

and the dominated convergence theorem, we obtain that the expresssion in (7.9) converges to 0 as $y_1 \to x_1$.

Finally, let $t, h \in \mathbb{R}_+$, $x_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}^*$. As in (6.19) and (6.20), (7.7) leads to

$$\begin{aligned} &\mathbb{E}((X(t+h, x_1, x_2) - X(t, x_1, x_2))^2) \\ &\quad = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t dr \; (\mathcal{F}_1 G(r+h, -\xi_1, x_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2 \qquad (7.10) \\ &\qquad + \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^h dq \; \mathcal{F}_1 G(q, -\xi_1, x_2)^2. \qquad (7.11) \end{aligned}$$

Using the same technique as above for the continuity in x_2, we show that (7.10) converges to 0 as $h \to 0$. First, note that the integrand converges to 0 for all r in $[0, t]$ such that $r \neq t - |x_2|$, and that it is bounded on $[0, t] \times \mathbb{R}^{d-1} \times \mathbb{R}$, so by the dominated convergence theorem,

$$\int_0^t dr \; (\mathcal{F}_1 G(r+h, -\xi_1, x_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2 \underset{h \to 0}{\to} 0.$$

But since for all $h \le h_0$,

$$\begin{aligned} &\int_0^t dr \; (\mathcal{F}_1 G(r+h, -\xi_1, x_2) - \mathcal{F}_1 G(r, -\xi_1, x_2))^2 \\ &\quad \le 2\left(\int_0^t dr \; \mathcal{F}_1 G(r+h, -\xi_1, x_2)^2 + \int_0^t dr \; \mathcal{F}_1 G(r, -\xi_1, x_2))^2\right) \\ &\quad \le 4 \int_0^{t+h_0} dr \; \mathcal{F}_1 G(r, -\xi_1, x_2)^2 \\ &\quad \le \frac{4\, C_5(t+h_0)}{\sqrt{1+|\xi_1|^2}} \operatorname{arccosh}\left(\frac{t+h_0}{|x_2|}\right) \; 1_{\{|x_2| < t+h_0\}}, \end{aligned}$$

by Lemma 7.1.2, we obtain that the expression (7.10) converges to 0 as $h \to 0$ by Assumption B_0 and the dominated convergence theorem. On the other hand, Lemma 7.1.1 implies that for all $h \le h_0$,

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^h dq \; \mathcal{F}_1 G(q, -\xi_1, x_2)^2 \\ &\quad \le \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^h dq \; \frac{C_4(h_0)}{\sqrt{1+|\xi_1|^2}} \; \frac{1}{\sqrt{q^2 - x_2^2}} \; 1_{\{|x_2| < q\}} \\ &\quad \le C_4(h_0) \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1+|\xi_1|^2}} \operatorname{arccosh}\left(\frac{h}{|x_2|}\right) \; 1_{\{|x_2| < h\}}, \end{aligned}$$

so by Assumption B_0, the integral in (7.11) converges also to 0 as $h \to 0$, and this shows the right-continuity in t of the process X (in $L^2(\Omega)$). The left-continuity follows in the same way as in the proof of Lemma 6.3.3, and this completes the proof. □

Lemma 7.2.4. *Let u be the solution of equation (6.5) (with $k = d-1$). Under Assumption B_0, the process X defined in Lemma 7.2.2 satisfies*

$$\langle u(t), \varphi\rangle = \int_{\mathbb{R}^d} dx\; X(t,x)\; \varphi(x), \qquad \mathbb{P} - a.s.,$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that supp $\varphi \subset \mathbb{R}^{d-1} \times \mathbb{R}^*$.

Proof. By Lemma 7.2.2 and Remark 7.2.3, the integral on the right-hand side of the above equation is well defined, since supp $\varphi \subset \mathbb{R}^{d-1} \times \mathbb{R}^*$. To show that both sides are equal $\mathbb{P}$-a.s., we proceed as in the proof of Lemma 6.3.5. By (6.7) and (6.13), we obtain that

$$\begin{aligned}
\mathbb{E}(|\langle u(t),\varphi\rangle|^2) &= \mathbb{E}\left(\left|\int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dx_1)\; (G(t-s)*\varphi)(x_1,0)\right|^2\right)\\
&= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\; |\mathcal{F}_1(G(t-s)*\varphi)(\xi_1,0)|^2.
\end{aligned}$$

Since $\mathcal{F}_1 = \mathcal{F}_2^{-1}\,\mathcal{F}$ and $\mathcal{F}(G*H) = \mathcal{F}G \cdot \mathcal{F}H$, we can write that

$$\begin{aligned}
\mathcal{F}_1(G(t-s)*\varphi)(\xi_1,0) &= \mathcal{F}_2^{-1}(\mathcal{F}G(t-s)\cdot\mathcal{F}\varphi)(\xi_1,0)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2\; \mathcal{F}G(t-s,\xi_1,\xi_2)\; \mathcal{F}\varphi(\xi_1,\xi_2),
\end{aligned} \tag{7.12}$$

where we have used (4.1), so

$$\mathbb{E}(|\langle u(t),\varphi\rangle|^2) = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\int_0^t ds\left|\frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2\; \mathcal{F}G(t-s,\xi_1,\xi_2)\;\mathcal{F}\varphi(\xi_1,\xi_2)\right|^2. \tag{7.13}$$

On the other hand, by Fubini's theorem and (7.7),

$$\begin{aligned}
&\mathbb{E}\left(\left|\int_{\mathbb{R}^d} dx\; X(t,x)\;\varphi(x)\right|^2\right)\\
&= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\; \mathbb{E}(X(t,x)\; X(t,y))\; \varphi(x)\;\overline{\varphi(y)}\\
&= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\int_0^t ds\left|\int_{\mathbb{R}^k} dx_1 \int_{\mathbb{R}^{d-k}} dx_2\; \mathcal{F}_1G(t-s,-\xi_1,x_2)\;\chi_{\xi_1}(x_1)\;\varphi(x_1,x_2)\right|^2.
\end{aligned} \tag{7.14}$$

Using now the definitions of $\mathcal{F}_1$ and $\mathcal{F}_2$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^k} dx_1 \int_{\mathbb{R}^{d-k}} dx_2\; \mathcal{F}_1G(t-s,-\xi_1,x_2)\;\chi_{\xi_1}(x_1)\;\varphi(x_1,x_2)\\
&= \int_{\mathbb{R}} dx_2\; \mathcal{F}_1G(t-s,-\xi_1,x_2)\;\; \mathcal{F}_1\varphi(\xi_1,x_2)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2\; \mathcal{F}G(t-s,-\xi_1,-\xi_2)\;\mathcal{F}\varphi(\xi_1,\xi_2),
\end{aligned} \tag{7.15}$$

which is equal to (7.12), by symmetry of $\mathcal{F}G$ in ξ, so (7.13) and (7.14) are equal. Following the proof of Lemma 6.3.5, it remains to compute, using Fubini's theorem and (6.13),

$$\begin{aligned}
&\mathbb{E}\left(\langle u(t), \varphi \rangle \cdot \overline{\int_{\mathbb{R}^d} dx\ X(t,x)\ \varphi(x)} \right) \\
&= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ \Bigg(\mathcal{F}_1(G(t-s) * \varphi)(\xi_1, 0) \\
&\qquad \cdot \overline{\int_{\mathbb{R}^d} dx\ \mathcal{F}_1 G(t-s, -\xi_1, x_2)\ \chi_{\xi_1}(x_1)\ \varphi(x)} \Bigg).
\end{aligned}$$

Using calculations (7.12) and (7.15), we obtain that this last expression is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ \Bigg(\frac{1}{2\pi} \int_{\mathbb{R}} d\xi_2\ \mathcal{F}G(t-s, \xi_1, \xi_2)\ \mathcal{F}\varphi(\xi_1, \xi_2) \\
&\qquad \cdot \overline{\frac{1}{2\pi} \int_{\mathbb{R}} d\xi_2\ \mathcal{F}G(t-s, -\xi_1, -\xi_2)\ \mathcal{F}\varphi(\xi_1, \xi_2)} \Bigg).
\end{aligned}$$

which is also equal to (7.13) and (7.14). This completes the proof. □

With these three lemmas in hand, we can now prove the following theorem, which shows moreover that Assumption B_0 is optimal, as already mentioned in Remark 6.4.5 in the preceding chapter.

Theorem 7.2.5. *Let u be the solution of equation (6.5)(with $k = d-1$). There exists a square integrable real-valued process $X = \{\ X(t, x_1, x_2),\ (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*\}$ such that the map $(t, x_1, x_2) \mapsto X(t, x_1, x_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ to $L^2(\Omega)$ and*

$$\langle u(t), \varphi \rangle = \int_{\mathbb{R}^d} dx\ X(t,x)\ \varphi(x), \qquad \mathbb{P}-a.s.,$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that supp $\varphi \subset \mathbb{R}^{d-1} \times \mathbb{R}^*$ if only if Assumption B_0 is satisfied. Moreover, when X exists, it is a centered Gaussian process whose covariance is given by formula (7.7).

Proof. This proof follows the same scheme as the proof of Proposition 6.3.6 in Section 6.3. The sufficiency of condition B_0 follows directly from the three preceding lemmas, so let us now care about the necessity: fix $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}$ such that $0 < |x_2| < t$ and let $\varphi^{(n)}_{x_1,x_2} = \delta_{(x_1,x_2)} * \psi_n \in \mathcal{S}(\mathbb{R}^d)$, where (ψ_n) is a sequence of non-negative and compactly supported approximations of δ_0 in $\mathbb{R}^d$. Since supp $\varphi^{(n)}_{x_1,x_2} \subset \mathbb{R}^{d-1} \times \mathbb{R}^*$ for n sufficiently large, the assumptions made on X and Fubini's theorem imply that

$$\begin{aligned}
\mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,x_2} \rangle|^2) &= \mathbb{E}\left(\left| \int_{\mathbb{R}^d} dy_1\ dy_2\ X(t, y_1, y_2)\ \varphi^{(n)}_{x_1,x_2}(y_1, y_2) \right|^2 \right) \\
&= \int_{\mathbb{R}^d} dy_1\ dy_2 \int_{\mathbb{R}^d} dz_1\ dz_2\ \mathbb{E}(X(t, y_1, y_2)\ X(t, z_1, z_2))\ \varphi^{(n)}_{x_1,x_2}(y_1, y_2)\ \overline{\varphi^{(n)}_{x_1,x_2}(z_1, z_2)} \\
&\underset{n\to\infty}{\to} \mathbb{E}(X(t, x_1, x_2)^2) < \infty. \qquad (7.16)
\end{aligned}$$

On the other hand, replacing φ by $\varphi^{(n)}_{x_1,x_2}$ in (7.13) gives

$$\mathbb{E}(|\langle u(t),\varphi^{(n)}_{x_1,x_2}\rangle|^2) = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds \left| \frac{1}{2\pi} \int_{\mathbb{R}} d\xi_2 \, \mathcal{F}G(t-s,\xi_1,\xi_2) \, \mathcal{F}\varphi^{(n)}_{x_1,x_2}(\xi_1,\xi_2) \right|^2.$$

Let us then compute

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbb{R}} d\xi_2 \, \mathcal{F}G(t-s,\xi_1,\xi_2) \, \mathcal{F}\varphi^{(n)}_{x_1,x_2}(\xi_1,\xi_2) \\ &= \int_{\mathbb{R}} dy_2 \, \mathcal{F}_1 G(t-s,\xi_1,-y_2) \, \mathcal{F}_1\varphi^{(n)}_{x_1,x_2}(\xi_1,y_2) \\ &= \int_{\mathbb{R}^d} dy_1 \int_{\mathbb{R}} dy_2 \, \mathcal{F}_1 G(t-s,\xi_1,-y_2) \, \chi_{\xi_1}(y_1) \, \varphi^{(n)}_{x_1,x_2}(y_1,y_2) \\ &\underset{n\to\infty}{\to} \mathcal{F}_1 G(t-s,\xi_1,-x_2) \, \chi_{x_1}(\xi_1), \end{aligned}$$

for all $(s,\xi_1) \in [0,t] \times \mathbb{R}^{d-1}$ such that $s \neq t - |x_2|$. Fatou's lemma and Lemma 7.1.3 then imply that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}(|\langle u(t),\varphi^{(n)}_{x_1,x_2}\rangle|^2) &\geq \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds \, \mathcal{F}_1 G(t-s,\xi_1,-x_2)^2 \\ &\geq C_6(t,x_2) \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1+|\xi_1|^2}}. \end{aligned}$$

Since the above limit exists and is finite by (7.16), Assumption B_0 is satisfied and this completes the proof. □

Remark 7.2.6. From estimate (7.5) in the proof of Lemma 7.2.1, one sees that under Assumption B_0 and for fixed $t \in \mathbb{R}_+$, there exists $C(t) > 0$ such that

$$\mathbb{E}(X(t,x_1,x_2)^2) \leq C(t) \operatorname{arccosh}\left(\frac{t}{|x_2|}\right) 1_{\{|x_2|<t\}} \underset{x_2\to 0}{\sim} \ln\left(\frac{1}{|x_2|}\right).$$

This estimate implies that

$$\int_{\mathbb{R}} dx_2 \, \mathbb{E}(X(t,x_1,x_2)^2) < \infty,$$

so by Fubini's theorem, the map $x_2 \mapsto X(t,x_1,x_2)$ belongs $\mathbb{P}-a.s.$ to $L^2(\mathbb{R})$, in concordance with Theorem 6.4.3 of the preceding chapter (in the case $k = d-1$ and $\beta = 0$). On the other hand, the behavior in $\ln(\frac{1}{|x_2|})$ is the reason why the process X is not defined on the hyperplane $x_2 = 0$. In the following section, we shall see that under a stronger assumption on the spectral measure μ, the process X can be defined also on the hyperplane $x_2 = 0$.

7.3 A stronger condition

By Theorem 7.2.5, Assumption B_0 only guarantees that the solution of equation (6.5) is a real-valued process X defined outside the hyperplane $x_2 = 0$. We are going to show here that the process X is defined on the whole space under the following slightly stronger condition on μ (which does not belong to the set of assumptions of the preceding chapter).

Assumption B_0'.

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)\ \ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}} < \infty.$$

Note that there is only an extra logarithmic factor in this assumption compared to Assumption B_0. At the end of this section, we give an example of a spectral measure μ which does not satisfy Assumption B_0' but satisfies Assumption B_0.

One can also notice that the situation is once again completely different in the case of the heat equation, since there never exists a real-valued solution defined on the whole space for this equation; see Chapter 9.

We will now prove the optimality of the above condition through the following two estimates, which are slightly more delicate to establish than Lemmas 7.1.2 and Lemmas 7.1.3, but give bounds valid for $x_2 = 0$.

Lemma 7.3.1. *For all $t > 0$, there exists $C_7(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq C_7(t)\ \frac{1+\ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}}, \qquad \forall \xi_1 \in \mathbb{R}^{d-1},\ x_2 \in \mathbb{R}.$$

Proof. If $|\xi_1|^2 \geq 2(a^2-b)+1$, then since

$$J_0(r)^2 \leq \frac{C}{\sqrt{1+r^2}}, \qquad \forall r \geq 0,$$

we obtain that

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 = \int_0^t ds\ \frac{e^{-2as}}{4}\ J_0\left(\sqrt{(|\xi_1|^2+b-a^2)\ (s^2-x_2^2)}\right)^2\ 1_{\{|x_2|\,<\,s\}}$$

$$\leq \frac{C\ e^{2a^- t}}{4} \int_{|x_2|}^t ds\ \frac{1}{\sqrt{1+(|\xi_1|^2+b-a^2)\ (s^2-x_2^2)}}\ 1_{\{|x_2|\,<\,t\}}.$$

Computing the integral gives

$$\frac{C\ e^{2a^- t}}{4\ \sqrt{|\xi_1|^2+b-a^2}}\ \ln\left(s+\sqrt{\frac{1}{|\xi_1|^2+b-a^2}+s^2-x_2^2}\right)\Bigg|_{s=|x_2|}^{s=t}\ 1_{\{|x_2|\,<\,t\}}$$

$$\leq \frac{\sqrt{2}\ C\ e^{2a^- t}}{4\ \sqrt{1+|\xi_1|^2}}\ \ln\left(\frac{t+\sqrt{\frac{1}{|\xi_1|^2+b-a^2}+t^2-x_2^2}}{|x_2|+\frac{1}{\sqrt{|\xi_1|^2+b-a^2}}}\right)\ 1_{\{|x_2|\,<\,t\}},$$

using (7.3). This last expression is maximum when $x_2 = 0$, in which case it is equal to

$$\frac{\sqrt{2}\ C\ e^{2a^- t}}{4\ \sqrt{1+|\xi_1|^2}}\ \left(\ln\left(\sqrt{|\xi_1|^2+b-a^2}\right)+\ln\left(t+\sqrt{\frac{1}{|\xi_1|^2+b-a^2}+t^2}\right)\right).$$

Morevoer, using (7.4) and the fact that $\frac{1}{|\xi_1|^2+b-a^2} \leq 2$, we obtain that

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{\sqrt{2}\ C\ e^{2a^- t}}{4\ \sqrt{1+|\xi_1|^2}} \left(\ln\left(\sqrt{1+|\xi_1|^2}\right) + \ln\left(\sqrt{1 \vee (b-a^2)}\right) + \ln\left(t+\sqrt{2+t^2}\right)\right).$$

If $2(a^2-b)+1 \geq 0$ and $a^2 - b \leq |\xi_1|^2 \leq 2(a^2-b)+1$, then since $J_0(r)^2 \leq 1$ for all $r \geq 0$, we get that

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \int_0^t ds\ \frac{e^{-2as}}{4}\ J_0\left(\sqrt{(|\xi_1|^2+b-a^2)\ (s^2-x_2^2)}\right)^2\ 1_{\{|x_2|<s\}} \\ &\leq \frac{e^{2a^- t}}{4}\ t.\end{aligned}$$

Finally, if $a^2 - b \geq 0$ and $|\xi_1|^2 \leq a^2 - b$, then since $I_0(r)^2 \leq C\ e^{2r}$ for all $r \geq 0$, we get that

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \int_0^t ds\ \frac{e^{-2as}}{4}\ I_0\left(\sqrt{(a^2-b-|\xi_1|^2)\ (s^2-x_2^2)}\right)^2\ 1_{\{|x_2|<s\}} \\ &\leq \frac{e^{2a^- t}}{4}\ C\ e^{2\sqrt{a^2-b}\,t}\ t,\end{aligned}$$

and the proof ends as the proof of Lemma 5.4.1. □

Lemma 7.3.2. *For all $t > 0$, there exist $C_8(t)$, $C_8'(t)$ and $R(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,0)^2 \geq \frac{C_8(t)\ \ln\left(\sqrt{1+|\xi_1|^2}\right) - C_8'(t)}{\sqrt{1+|\xi_1|^2}}, \qquad \forall \xi_1 \in \mathbb{R}^{d-1} \text{ with } |\xi_1| \geq R(t).$$

Proof. Let $R(t)^2 = 2(a^2-b) + (\frac{\pi^2}{t^2} \vee 1)$ and $|\xi_1| \geq R(t)$. We then compute

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,0)^2 &= \int_0^t ds\ \frac{e^{-2as}}{4}\ J_0\left(s\ \sqrt{|\xi_1|^2+b-a^2}\right)^2 \\ &\geq \frac{e^{-2a^+ t}}{4} \int_0^t ds\ J_0\left(s\ \sqrt{|\xi_1|^2+b-a^2}\right)^2 \\ &= \frac{e^{-2a^+ t}}{4\ \sqrt{|\xi_1|^2+b-a^2}} \int_0^{t\sqrt{|\xi_1|^2+b-a^2}} dr\ J_0(r)^2,\end{aligned}$$

by the change of variable $r = s\sqrt{|\xi_1|^2+b-a^2}$. Using now (7.4), we obtain that

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,0)^2 &\geq \frac{e^{-2a^+ t}}{4\ \sqrt{1 \vee (b-a^2)}\ \sqrt{1+|\xi_1|^2}} \int_1^{t\sqrt{|\xi_1|^2+b-a^2}} dr\ J_0(r)^2 \\ &\geq \frac{e^{-2a^+ t}}{4\ \sqrt{1 \vee (b-a^2)}\ \sqrt{1+|\xi_1|^2}} \left(\frac{\ln\left(t\ \sqrt{|\xi_1|^2+b-a^2}\right)}{\pi} - C\right),\end{aligned}$$

by Lemma B.2.2. Using (7.3), we moreover have

$$\ln\left(t\ \sqrt{|\xi_1|^2+b-a^2}\right) \geq \ln\left(\sqrt{1+|\xi_1|^2}\right) + \ln\left(\frac{t}{\sqrt{2}}\right),$$

so the conclusion follows. □

We can now state the theorem.

Theorem 7.3.3. *Let u be the solution of equation (6.5) (with $k = d-1$). There exists a square integrable real-valued process $X = \{X(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ such that the map $(t,x) \mapsto X(t,x)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^d$ to $L^2(\Omega)$ and*

$$\langle u(t), \varphi \rangle = \int_{\mathbb{R}^d} dx\, X(t,x)\, \varphi(x), \qquad \mathbb{P}-a.s.,\ \forall t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d),$$

if and only if Assumption B_0' is satisfied. Moreover, when X exists, it is a centered Gaussian process whose covariance is given by formula (7.7).

Proof. The proof is similar to that of Theorem 7.2.5. Let us first show the sufficiency of Assumption B_0', considering what needs to be modified in Lemmas 7.2.1, 7.2.2 and 7.2.4.

In Lemma 7.2.1, we simply use Lemma 7.3.1 and Assumption B_0' instead of Lemma 7.1.2 and Assumption B_0 in order to estimate $\|\phi_{t,x_1,x_2}\|_t$, which gives us the finiteness of this expression for all $(t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}$.

For Lemma 7.2.2, we need some slightly different estimates of the L^2-increments of the process. Let us first consider $x_2,\ y_2 \in \mathbb{R}$. We have, following the proof of this lemma,

$$\begin{aligned}
&\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \mathbb{E}((X(t,x_1,y_2) - X(t,x_1,x_2))^2) \\
&\quad = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^T dr\ (\mathcal{F}_1 G(r,-\xi_1,y_2) - \mathcal{F}_1 G(r,-\xi_1,x_2))^2.
\end{aligned}$$

Using twice the dominated convergence theorem as in the proof of Lemma 7.2.2, jointly with Lemma 7.3.1 and Assumption B_0', we obtain that the above expression converges to 0 as $y_2 \to x_2$.

Now, let $x_1,\ y_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}$. As above, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}((X(t,y_1,x_2) - X(t,x_1,x_2))^2) \\
&\quad = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^T dr\ \mathcal{F}_1 G(r,-\xi_1,x_2)^2\ 2\,(1-\cos(\xi_1\cdot(y_1-x_1))).
\end{aligned}$$

Once again, using twice the dominated convergence theorem joinlty with Lemma 7.3.1 and Assumption B_0', we obtain that the above expression converges to 0 as $y_1 \to x_1$.

Finally, let $t,h \in \mathbb{R}_+$, $x_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}$. Then

$$\begin{aligned}
&\mathbb{E}((X(t+h,x_1,x_2) - X(t,x_1,x_2))^2) \\
&\quad = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t dr\ (\mathcal{F}_1 G(r+h,-\xi_1,x_2) - \mathcal{F}_1 G(r,-\xi_1,x_2))^2 && (7.17) \\
&\qquad + \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^h dq\ \mathcal{F}_1 G(q,-\xi_1,x_2)^2. && (7.18)
\end{aligned}$$

Using the same arguments as above, we obtain that (7.17) converges to 0 as $h \to 0$. For the second term, note that as $\mathcal{F}_1 G$ is bounded on $[0,h] \times \mathbb{R}^{d-1} \times \mathbb{R}$,

$$\int_0^h dq\, \mathcal{F}_1 G(q, -\xi_1, x_2)^2 \underset{h\to 0}{\to} 0, \qquad \forall \xi_1 \in \mathbb{R}^{d-1}.$$

Moreover, by Lemma 7.3.1,

$$\int_0^h dq\, \mathcal{F}_1 G(q, -\xi_1, x_2)^2 \leq C_7(h_0)\, \frac{1 + \ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}},$$

for all $h \leq h_0$. So the dominated convergence theorem allows us to conclude that the process X is L^2-right-continous in t, and an argument similar to that of the proof of Lemma 6.3.3 allows us to to prove the left-continuity. Summing up these results gives us the L^2-continuity of the process X on $\mathbb{R}_+ \times \mathbb{R}^d$, then the existence of a jointly measurable modification.

The proof of Lemma 7.2.4 remains unchanged, except that the condition supp $\varphi \subset \mathbb{R}^{d-1} \times \mathbb{R}^*$ disappears. This completes the proof of the sufficiency.

In order to prove the necessity, let us also follow the proof of Theorem 7.2.5. Assuming that the process X exists, we therefore have

$$\infty > \mathbb{E}(X(t, x_1, 0)^2) = \lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,0}\rangle|^2, \tag{7.19}$$

where $\varphi^{(n)}_{x_1,0} \underset{n\to\infty}{\to} \delta_{x_1,0}$ in $\mathcal{S}'(\mathbb{R}^d)$. Using (7.13), we obtain

$$\mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,0}\rangle|^2) = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds \left| \frac{1}{2\pi} \int_{\mathbb{R}} d\xi_2\, \mathcal{F}G(t-s, \xi_1, \xi_2)\, \mathcal{F}\varphi^{(n)}_{x_1,0}(\xi_1, \xi_2) \right|^2.$$

So by the same calculations as in the proof of Theorem 7.2.5 and Fatou's lemma, we have

$$\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,0}\rangle|^2 \geq \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\, \mathcal{F}_1 G(t-s, \xi_1, 0)^2.$$

But since by Lemma 7.3.2,

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\, \mathcal{F}_1 G(t-s, \xi_1, 0)^2 \; &\geq \; \int_{B_1(0,R(t))^c} \mu(d\xi_1) \int_0^t ds\, \mathcal{F}_1 G(t-s, \xi_1, 0)^2 \\ &\geq \; \int_{B_1(0,R(t))^c} \mu(d\xi_1)\, \frac{C_8(t)\, \ln\left(\sqrt{1+|\xi_1|^2}\right) - C_8'(t)}{\sqrt{1+|\xi_1|^2}}, \end{aligned}$$

we obtain, using (7.19) and the fact that $\mu(B_1(0, R(t))) < \infty$,

$$\int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\, \frac{C_8(t)\, \ln\left(\sqrt{1+|\xi_1|^2}\right) - C_8'(t)}{\sqrt{1+|\xi_1|^2}} < \infty.$$

Now we use Theorem 7.2.5 which tells us that under the assumptions made in the present theorem, Assumption B_0 is satisfied, therefore,

$$C_8(t) \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \frac{\ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}} = \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \frac{C_8(t)\ln\left(\sqrt{1+|\xi_1|^2}\right) - C_8'(t)}{\sqrt{1+|\xi_1|^2}} + C_8'(t) \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1+|\xi_1|^2}} < \infty,$$

so Assumption B_0' is satisfied, and this completes the proof. □

This proves that under condition B_0', the weak solution u of equation (6.2) does not explode near $x_2 = 0$. Let us now give an example of a spectral measure μ which satisfies B_0 but not B_0', and for which there is therefore an explosion near $x_2 = 0$, by the necessity of condition B_0'. We consider the case $d = 2$ (that is, the case where μ is a measure on the real line) and describe μ by its density ϕ given by

$$\phi(r) = \begin{cases} 3 - \frac{2r}{e}, & \text{if } r \in [0, e[, \\ \frac{1}{\ln(r)^2} & \text{if } r \in [e, \infty[, \end{cases}$$

and $\phi(r) = \phi(-r)$ for $r < 0$. One can easily check that

$$\frac{\phi(r)}{\sqrt{1+r^2}} \underset{r\to\infty}{\sim} \frac{1}{\ln(r)^2\, r}, \quad \text{so} \quad \int_{\mathbb{R}} dr\, \frac{\phi(r)}{\sqrt{1+r^2}} < \infty.$$

On the other hand,

$$\frac{\phi(r)\, \ln(\sqrt{1+r^2})}{\sqrt{1+r^2}} \underset{r\to\infty}{\sim} \frac{1}{\ln(r)\, r}, \quad \text{so} \quad \int_{\mathbb{R}} dr\, \frac{\phi(r)\, \ln(\sqrt{1+r^2})}{\sqrt{1+r^2}} = \infty.$$

Let us now check that the corresponding covariance $\Gamma = \mathcal{F}_1\mu$ satisfies all the required assumptions. Clearly, μ is a non-negative tempered Borel measure on $\mathbb{R}$, so Γ is a tempered non-negative definite distribution by the Bochner-Schwartz theorem 4.3.1. It remains to show that Γ is a measure on $\mathbb{R}$. For this, let us note that ϕ is decreasing and convex on $[0, \infty[$, so by Polya's criterion (see for example [23, §2.3.d]), ϕ is a (symmetric) non-negative definite function on $\mathbb{R}$. By the classical Bochner theorem, this implies that Γ is a non-negative finite measure on $\mathbb{R}$. We therefore have constructed a relevant example.

7.4 Hölder-continuity of the solution

In this section, we show that when β is positive, Assumption B_β of the preceding chapter implies a stronger regularity of the solution than the one obtained in that chapter. Namely, we can show here Hölder-regularity of the solution. The techniques that we use are similar to those used in [54] for the solution of the hyperbolic equation driven by spatially homogeneous noise.

For the clarity of the calculations, we assume in this section that the coefficients a and b of equation (6.5) are both equal to 0 and that the space dimension d belongs to $\{2, ..., 5\}$ (because of the preceding restriction). Let then $\beta \in \left]0, \frac{1}{2}\right[$ and suppose that **Assumption** B_β of the preceding chapter is satisfied, which can be written when $k = d-1$ as

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}} < \infty.$$

Note that this condition is always stronger than B_0 *and* B_0', and that when β tends to $\frac{1}{2}$, it looks more and more like "μ is a finite measure", as it was the case for the condition A_α of the preceding chapter, with α tending to 1.

We will now prove that under this assumption, the process X admits a modification which is $\mathbb{P} - a.s.$ locally Hölder-continuous outside the hyperplane $x_2 = 0$, with exponent $\gamma < \beta \wedge \frac{1}{4}$. For this, we need the following two lemmas.

Lemma 7.4.1. *Fix $\beta \in \left]0, \frac{1}{2}\right[$. For all $T > 0$ and $R > \varepsilon > 0$, there exists $C_9(T, R, \varepsilon) > 0$ such that*

$$\int_0^t ds\ (\mathcal{F}_1 G(s, \xi_1, y_2) - \mathcal{F}_1 G(s, \xi_1, x_2))^2 \leq C_9(T, R, \varepsilon)\ \frac{|y_2 - x_2|^{2\beta\wedge\frac{1}{2}}}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}},$$

for all $t \in [0, T]$, $\xi_1 \in \mathbb{R}^{d-1}$ and $\varepsilon \leq |x_2|,\ |y_2| \leq R$.

Proof. Let us first recall that we have assumed that $a = b = 0$, so the Fourier transform of G in the coordinate x_1 simply reads (see (7.1))

$$\mathcal{F}_1 G(t, \xi_1, x_2) = \frac{1}{2}\ J_0\left(|\xi_1|\ \sqrt{t^2 - x_2^2}\right)\ 1_{\{|x_2| < t\}}, \qquad \forall (t, \xi_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \in \mathbb{R}.$$

Let us fix $t \in [0, T]$, $\xi_1 \in \mathbb{R}^{d-1}$, $\varepsilon \leq |x_2|,\ |y_2| \leq R$ and assume, without loss of generality, that $\varepsilon \leq x_2 \leq y_2 \leq R$. Let us then compute

$$\int_0^t ds\ (\mathcal{F}_1 G(s, \xi_1, y_2) - \mathcal{F}_1 G(s, \xi_1, x_2))^2$$
$$= \frac{1}{4}\int_0^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right)\ 1_{\{y_2 < s\}} - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\ 1_{\{x_2 < s\}}\right)^2. \tag{7.20}$$

Adding and substracting the term $J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\ 1_{\{y_2 < s\}}$ inside the parentheses, we obtain that (7.20) is less than or equal to

$$\frac{1}{2}\ 1_{\{y_2 < t\}} \int_{y_2}^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^2 \tag{7.21}$$
$$+ \frac{1}{2}\int_{x_2}^{y_2} ds\ J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)^2. \tag{7.22}$$

We now need to bound these two terms. We begin with (7.22). Since $J_0(r)^2 \leq \frac{C}{r}$ for all $r > 0$, we obtain that when $|\xi_1| \geq 1$,

$$\begin{aligned}
&\int_{x_2}^{y_2} ds\ J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)^2 \\
&\leq \frac{C}{|\xi_1|} \int_{x_2}^{y_2} ds\ \frac{1}{\sqrt{s^2 - x_2^2}} = \frac{C}{|\xi_1|} \int_{x_2}^{y_2} ds\ \frac{1}{\sqrt{(s + x_2)\ (s - x_2)}} \\
&\leq \frac{C}{|\xi_1|\ \sqrt{2x_2}} \int_{x_2}^{y_2} ds\ \frac{1}{\sqrt{(s - x_2)}} \leq \frac{2C}{\sqrt{2\varepsilon}}\ \frac{\sqrt{y_2 - x_2}}{|\xi_1|},
\end{aligned}$$

since $s + x_2 \geq 2x_2 \geq 2\varepsilon$. On the other hand, when $|\xi_1| \leq 1$,

$$\int_{x_2}^{y_2} ds\ J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)^2 \leq y_2 - x_2 \leq \sqrt{2R}\ \sqrt{y_2 - x_2},$$

since $J_0(r)^2 \leq 1$ for all $r \geq 0$ and $0 \leq x_2 \leq y_2 \leq R$. So there exists $C(R, \epsilon) > 0$ such that (7.22) is less than or equal to

$$C(R, \varepsilon)\ \frac{\sqrt{y_2 - x_2}}{\sqrt{1 + |\xi_1|^2}}.$$

We now turn to (7.21). If $|\xi_1| \geq 1$, then since $J_0'(r) = -J_1(r)$ for all $r \geq 0$, where J_1 is the first order regular Bessel function of the first kind (see Appendix B), we have

$$\begin{aligned}
&\left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^2 \\
&= \left(\int_{|\xi_1|\ \sqrt{s^2 - y_2^2}}^{|\xi_1|\ \sqrt{s^2 - x_2^2}} dr\ J_1(r)\right)^{2\beta} \left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^{2(1-\beta)}.
\end{aligned}$$

Using the fact that $|J_1(r)| \leq \frac{C}{\sqrt{r}}$ for all $r > 0$, we obtain

$$\left(\int_{|\xi_1|\ \sqrt{s^2 - y_2^2}}^{|\xi_1|\ \sqrt{s^2 - x_2^2}} dr\ J_1(r)\right)^{2\beta} \leq (2C)^{2\beta}\ |\xi_1|^\beta\ \left((s^2 - x_2^2)^{\frac{1}{4}} - (s^2 - y_2^2)^{\frac{1}{4}}\right)^{2\beta}.$$

Since $x_2 \leq y_2$ and for all $v \geq u \geq 0$,

$$v^{\frac{1}{4}} - u^{\frac{1}{4}} = \frac{\sqrt{v} - \sqrt{u}}{v^{\frac{1}{4}} + u^{\frac{1}{4}}} \leq \frac{v - u}{2u^{\frac{1}{4}}\ (\sqrt{v} + \sqrt{u})} \leq \frac{v - u}{4u^{\frac{3}{4}}},$$

we get that

$$\begin{aligned}
\left(\int_{|\xi_1|\ \sqrt{s^2 - y_2^2}}^{|\xi_1|\ \sqrt{s^2 - x_2^2}} dr\ J_1(r)\right)^{2\beta} &\leq (2C)^{2\beta}\ |\xi_1|^\beta\ \frac{(y_2^2 - x_2^2)^{2\beta}}{4(s^2 - y_2^2)^{\frac{3\beta}{2}}} \\
&\leq (2CR)^{2\beta}\ |\xi_1|^\beta\ \frac{(y_2 - x_2)^{2\beta}}{(s^2 - y_2^2)^{\frac{3\beta}{2}}},
\end{aligned}$$

where we have used the fact that $y_2^2 - x_2^2 = (y_2 + x_2)\ (y_2 - x_2) \leq 2R\ (y_2 - x_2)$. On the other hand, since $J_0(r)^2 \leq \frac{C}{r}$ for all $r > 0$ and $y_2 \geq x_2$,

$$\begin{aligned}
&\left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^{2(1-\beta)}\\
&\leq \left(2\ J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right)^2 + 2\ J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)^2\right)^{1-\beta}\\
&\leq \frac{(4C)^{1-\beta}}{|\xi_1|^{1-\beta}\ (s^2 - y_2^2)^{\frac{1-\beta}{2}}}.
\end{aligned}$$

Combining the above estimates gives

$$\begin{aligned}
&1_{\{y_2\ <\ t\}} \int_{y_2}^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^2\\
&\leq\ C\ R^{2\beta}\ 1_{\{y_2\ <\ t\}} \int_{y_2}^t ds\ \frac{1}{(s^2 - y_2^2)^{\frac{1}{2}+\beta}}\ \frac{(y_2 - x_2)^{2\beta}}{|\xi_1|^{1-2\beta}}.
\end{aligned}$$

Since $\beta \in\ ]0, \frac{1}{2}[$,

$$\sup_{t\in[0,T],y_2\in[\varepsilon,R]} 1_{\{y_2\ <\ t\}} \int_{y_2}^t ds\ \frac{1}{(s^2 - y_2^2)^{\frac{1}{2}+\beta}} < \infty, \tag{7.23}$$

so we have obtained the desired bound for (7.21) when $|\xi_1| \geq 1$. On the other hand, when $|\xi_1| \leq 1$,

$$\begin{aligned}
&\left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^2\\
&= \left(\int_{|\xi_1|\ \sqrt{s^2-y_2^2}}^{|\xi_1|\ \sqrt{s^2-x_2^2}} dr\ J_1(r)\right)^2 \leq \left(\sqrt{s^2 - x_2^2} - \sqrt{s^2 - y_2^2}\right)^2,
\end{aligned}$$

since $J_1(r)^2 \leq 1$ for all $r \geq 0$ and $|\xi_1| \leq 1$. Moreover,

$$\begin{aligned}
\sqrt{s^2 - x_2^2} - \sqrt{s^2 - y_2^2}\ &\leq\ \frac{y_2^2 - x_2^2}{2\sqrt{s^2 - y_2^2}} \leq R\ \frac{y_2 - x_2}{\sqrt{s^2 - y_2^2}}\\
&\leq\ R\ (2R)^{1-2\beta}\ \frac{(y_2 - x_2)^{2\beta}}{\sqrt{s^2 - y_2^2}},
\end{aligned}$$

so we obtain that

$$\begin{aligned}
&1_{\{y_2\ <\ t\}} \int_{y_2}^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{s^2 - y_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2 - x_2^2}\right)\right)^2\\
&\leq\ R\ (2R)^{1-2\beta}\ 1_{\{y_2\ <\ t\}} \int_{y_2}^t ds\ \frac{1}{\sqrt{s^2 - y_2^2}}\ (y_2 - x_2)^{2\beta}.
\end{aligned}$$

Since

$$\sup_{t\in[0,T],y_2\in[\varepsilon,R]} 1_{\{y_2\ <\ t\}} \int_{y_2}^t ds\ \frac{1}{\sqrt{s^2 - y_2^2}} < \infty, \tag{7.24}$$

we conclude that there exists $C(T,R,\varepsilon) > 0$ such that (7.21) is less than or equal to

$$C(T,R,\varepsilon)\ \frac{(y_2-x_2)^{2\beta}}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}}.$$

Combining the two bounds obtained for (7.21) and (7.22) gives the desired result. □

Lemma 7.4.2. *Fix $\beta \in\,]0,\frac{1}{2}[$. For all $T > 0$ and $R > \varepsilon > 0$, there exists $C_{10}(T,R,\varepsilon) > 0$ such that*

$$\int_0^t ds\ (\mathcal{F}_1 G(s+h,\xi_1,x_2) - \mathcal{F}_1 G(s,\xi_1,x_2))^2 \leq C_{10}(T,R,\varepsilon)\ \frac{h^{2\beta\wedge\frac{1}{2}}}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}},$$

for all $t\in[0,T]$, $h \in [0,T-t]$, $\xi_1 \in \mathbb{R}^{d-1}$ and $\varepsilon \leq |x_2| \leq R$.

Proof. The proof is quite similar to the preceding one. Fix $t \in [0,T]$, $h \in [0,T-t]$, $\xi_1 \in \mathbb{R}^{d-1}$, $\varepsilon \leq x_2 \leq R$ and compute

$$\begin{aligned}
&\int_0^t ds\ (\mathcal{F}_1 G(s+h,\xi_1,x_2) - \mathcal{F}_1 G(s,\xi_1,x_2))^2 \\
&= \frac{1}{4}\int_0^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{(s+h)^2-x_2^2}\right)\ 1_{\{x_2<s+h\}} - J_0\left(|\xi_1|\ \sqrt{s^2-x_2^2}\right)\ 1_{\{x_2<s\}}\right)^2 \\
&\leq \frac{1}{2}\, 1_{\{x_2\,<\,t\}}\int_{x_2}^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{(s+h)^2-x_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2-x_2^2}\right)\right)^2 \qquad (7.25)\\
&\quad +\frac{1}{2}\int_{x_2-h}^{x_2} ds\ J_0\left(|\xi_1|\ \sqrt{(s+h)^2-x_2^2}\right)^2. \qquad (7.26)
\end{aligned}$$

Let us first bound the term in (7.26), using arguments similar to those used to bound (7.22). When $|\xi_1| \geq 1$,

$$\begin{aligned}
\int_{x_2-h}^{x_2} ds\ J_0\left(|\xi_1|\ \sqrt{(s+h)^2-x_2^2}\right)^2 &\leq \frac{C}{|\xi_1|}\int_{x_2-h}^{x_2} ds\ \frac{1}{\sqrt{(s+h+x_2)\ (s+h-x_2)}}\\
&\leq \frac{2C}{\sqrt{2\varepsilon}}\ \frac{\sqrt{h}}{|\xi_1|},
\end{aligned}$$

since $s+h+x_2 \geq 2x_2 \geq 2\varepsilon$. On the other hand, when $|\xi_1| \leq 1$,

$$\int_{x_2-h}^{x_2} ds\ J_0\left(|\xi_1|\ \sqrt{s^2-x_2^2}\right)^2 \leq h \leq \sqrt{T}\ \sqrt{h},$$

so there exists $C(T,\varepsilon) > 0$ such that (7.26) is less than or equal to

$$C(T,\varepsilon)\ \frac{\sqrt{h}}{\sqrt{1+|\xi_1|^2}}.$$

For the term (7.25), an argument analogous to that used for (7.21) gives, when $|\xi_1| \geq 1$,

$$\begin{aligned}
&1_{\{x_2\,<\,t\}}\int_{x_2}^t ds\ \left(J_0\left(|\xi_1|\ \sqrt{(s+h)^2-x_2^2}\right) - J_0\left(|\xi_1|\ \sqrt{s^2-x_2^2}\right)\right)^2\\
&\leq\ C\ T^{2\beta}\ 1_{\{x_2\,\leq\,t\}}\int_{x_2}^t ds\ \frac{1}{(s^2-x_2^2)^{\frac{1}{2}+\beta}}\ \frac{h^{2\beta}}{|\xi_1|^{1-2\beta}},
\end{aligned}$$

and when $|\xi_1| \leq 1$,

$$1_{\{x_2 < t\}} \int_{x_2}^{t} ds \, \left(J_0\left(|\xi_1| \sqrt{(s+h)^2 - x_2^2}\right) - J_0\left(|\xi_1| \sqrt{s^2 - x_2^2}\right)\right)^2$$
$$\leq T \, (2T)^{1-2\beta} \, 1_{\{x_2 \leq t\}} \int_{x_2}^{t} ds \, \frac{1}{\sqrt{s^2 - x_2^2}} \, h^{2\beta}.$$

(7.23) and (7.24) then allow us to conclude that there exists $C(T, R, \varepsilon) > 0$ such that (7.25) is less than or equal to

$$C(T, R, \varepsilon) \, \frac{h^{2\beta}}{(1 + |\xi_1|^2)^{\frac{1}{2} - \beta}},$$

and combining the two bounds obtained for (7.25) and (7.26) gives the result. □

Let us now state the theorem.

Theorem 7.4.3. *Let $\beta \in \,]0, \frac{1}{2}[$, let us make Assumption B_β and let X be the process defined in Theorem 7.3.3. There exists then a modification $\tilde{X}$ of the process X such that the map $(t, x_1, x_2) \mapsto \tilde{X}(t, x_1, x_2)$ is $\mathbb{P}$-a.s. locally Hölder-continuous on $\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ with exponent $\gamma < \beta \wedge \frac{1}{4}$, that is, for all $T > 0$ and $R > \varepsilon > 0$, there exists a $\mathbb{P} -$ a.s. positive random variable $\delta(\omega)$ and a constant $K(T, R, \varepsilon) > 0$ such that*

$$\mathbb{P}\left(\sup_{|u-v|<\delta, \; u,v \in B(T,R,\varepsilon),} \frac{|\tilde{X}(u) - \tilde{X}(v)|}{|u-v|^\gamma} \leq K(T, R, \varepsilon) \right) = 1,$$

where

$$B(T, R, \varepsilon) = [0, T] \times \mathbb{R}^{d-1} \times ([-R, -\varepsilon] \cup [\varepsilon, R])$$

and

$$|u| = \sqrt{t^2 + |x_1|^2 + x_2^2} \quad \text{for} \quad u = (t, x_1, x_2) \in B(T, R, \varepsilon).$$

Note that this theorem implies that if Assumption B_β is satisfied for any $\beta \in \,]0, \frac{1}{2}[$, then the modification $\tilde{X}$ of the process X is continuous outside the hyperplane $x_2 = 0$, and in particular on the set $|x_2| = t$, which was not clear a priori.

We point out that in the study of the heat equation, this question of regularity of the solution outside the hyperplane $x_2 = 0$ is answered in a simple manner through the regularizing property of the Green kernel: see Chapter 9.

Proof. We want to apply here the Kolmogorov continuity theorem, so we need first to study carefully the continuity in $L^2(\Omega)$ of the process X. Therefore, let $T > 0$ and $R > \varepsilon > 0$. We would like to estimate the L^2-increment

$$\mathbb{E}((X(t+h, y_1, y_2) - X(t, x_1, x_2))^2)$$

when $t \in [0,T]$, $h \in [0, T-t]$, $x_1,\ y_1 \in \mathbb{R}^{d-1}$ and $\varepsilon \leq |x_2|,\ |y_2| \leq R$. To this end, note that this increment is less than or equal to

$$\begin{aligned} &3\ (\mathbb{E}((X(t+h,y_1,y_2) - X(t,y_1,y_2))^2) \\ &+\mathbb{E}((X(t,y_1,y_2) - X(t,x_1,y_2))^2) \\ &+\mathbb{E}((X(t,x_1,y_2) - X(t,x_1,x_2))^2)). \end{aligned} \tag{7.27}$$

Let us consider the three terms separately, beginning by the last one. Using (7.7) and Lemma 7.4.1, we obtain that

$$\begin{aligned} &\mathbb{E}((X(t,x_1,y_2) - X(t,x_1,x_2))^2) \\ &= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ (\mathcal{F}_1 G(t-s,-\xi_1,y_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2 \\ &\leq C_9(T,R,\varepsilon) \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}}\ |y_2 - x_2|^{2\beta\wedge\frac{1}{2}}, \end{aligned}$$

the integral in ξ_1 being finite by Assumption B_β. For the second term in (7.27), we have by (7.7),

$$\begin{aligned} &\mathbb{E}((X(t,y_1,y_2) - X(t,x_1,y_2))^2) \\ &= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ \mathcal{F}_1 G(t-s,-\xi_1,y_2)^2\ |\chi_{y_1}(\xi_1) - \chi_{x_1}(\xi_1)|^2. \end{aligned}$$

Since

$$\begin{aligned} |\chi_{x_1}(\xi_1) - \chi_{y_1}(\xi_1)|^2 &\leq 4\ |\chi_{x_1}(\xi_1) - \chi_{y_1}(\xi_1)|^{2\beta} = 4\ \left| \int_{x_1\cdot\xi_1}^{y_1\cdot\xi_1} dr\ e^{ir} \right|^{2\beta} \\ &\leq 4\ |\xi_1|^{2\beta}\ |y_1 - x_1|^{2\beta}, \end{aligned}$$

we obtain by Lemma 7.1.2 that

$$\begin{aligned} &\mathbb{E}((X(t,y_1,y_2) - X(t,x_1,y_2))^2) \\ &\leq 4\ C_5(T)\ \text{arccosh}\left(\frac{t}{|y_2|}\right)\ 1_{\{|y_2| < t\}} \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}}\ |x_1 - y_1|^{2\beta} \\ &\leq 4\ C_5(T)\ \text{arccosh}\left(\frac{T}{\varepsilon}\right)\ 1_{\{\varepsilon < T\}} \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}}\ |x_1 - y_1|^{2\beta}. \end{aligned}$$

Finally, let us consider the first term in (7.27), which is equal to

$$\mathbb{E}((X(t+h,y_1,y_2) - X(t,y_1,y_2))^2)$$

$$= \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t dr\ (\mathcal{F}_1 G(r+h,-\xi_1,y_2) - \mathcal{F}_1 G(r,-\xi_1,y_2))^2 \tag{7.28}$$

$$+ \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^h dq\ \mathcal{F}_1 G(q,-\xi_1,y_2)^2, \tag{7.29}$$

by (7.10) and (7.11) in the proof of Lemma 7.2.2. By Lemma 7.4.2, (7.28) is now less than or equal to

$$C_{10}(T,R,\varepsilon) \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1+|\xi_1|^2)^{\frac{1}{2}-\beta}}\ h^{2\beta\wedge\frac{1}{2}}.$$

Moreover, $\mathcal{F}_1 G(q, \xi_1, -y_2) = 0$ when $q \le |y_2|$, so (7.29) is equal to 0 when $h \le \varepsilon$, and by Lemma 7.1.2, it is also bounded by

$$C(T, \varepsilon) = C_5(T) \ \text{arccosh}\left(\frac{T}{\varepsilon}\right) \ 1_{\{\varepsilon < T\}} \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{\sqrt{1 + |\xi_1|^2}},$$

so (7.29) is less than or equal to $C(T, \varepsilon) \ \frac{h}{\varepsilon}$. Summing up all these bounds gives the following result: there exists $C(T, R, \varepsilon) > 0$ such that

$$\begin{aligned} &\mathbb{E}((X(t+h, y_1, y_2) - X(t, x_1, x_2))^2) \\ &\le \ C(T, R, \varepsilon) \ \left(h^{2\beta \wedge \frac{1}{2}} + |y_1 - x_1|^{2\beta} + |y_2 - x_2|^{2\beta \wedge \frac{1}{2}}\right) \\ &\le \ 3\, C(T, R, \varepsilon) \ (h^2 + |y_1 - x_1|^2 + (y_2 - x_2)^2)^{\beta \wedge \frac{1}{4}}. \end{aligned}$$

Since X is a Gaussian process, this implies that for all $m \ge 1$, there exists $C^{(m)}(T, R, \varepsilon) > 0$ such that

$$\mathbb{E}((X(t+h, y_1, y_2) - X(t, x_1, x_2))^{2m}) \le C^{(m)}(T, R, \varepsilon) \ (h^2 + |y_1 - x_1|^2 + (y_2 - x_2)^2)^{(\beta \wedge \frac{1}{4})m}.$$

By the Kolmogorov continuity theorem (see for example [29, Problem 2.2.9]), we obtain that there exists a modification $\tilde{X}$ of X such that the map $(t, x_1, x_2) \mapsto \tilde{X}(t, x_1, x_2)$ is $\mathbb{P}$-a.s. locally Hölder-continuous on $\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ with exponent $\gamma < \beta \wedge \frac{1}{4}$. □

Moreover, one can notice that by the proof of the preceding theorem and under Assumption B_β, the process X admits a modification which is $\mathbb{P} - a.s.$ locally Hölder in x_1 with exponent $\gamma < \beta$, which is an improvement when $\beta > \frac{1}{4}$.

7.5 Reformulation of the conditions on the spectral measure

Following the scheme of Section 6.6, let us assume that the covariance Γ is non-negative and note that Assumption B_β is condition (4.9) with d replaced by $d-1$ and η replaced by $\frac{1}{2} - \beta$. Using then (4.10) and Proposition 4.4.1, we obtain that Assumption B_β is equivalent to

$$\int_{\mathbb{R}^{d-1}} \Gamma(dx_1) \ G_{d-1, \frac{1}{2} - \beta}(x_1) < \infty, \tag{7.30}$$

modulo the boundedness assumption of Proposition 4.4.1. As before, let us now make this last condition more explicit.

- If $d = 2$ and $\beta = 0$, then (7.30) is satisfied if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1) \ \ln\left(\frac{1}{|x_1|}\right) < \infty.$$

- If $d > 2$ or $\beta \in]0, \frac{1}{2}[$, then (7.30) is satisfied if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1) \ \frac{1}{|x_1|^{2\beta + d - 2}} < \infty.$$

In the case where $\Gamma(dx_1) = f(|x_1|)\, dx_1$, with f a continuous function on $]0,\infty[$, this implies that:

- If $d = 2$ and $\beta = 0$, then (7.30) is satisfied if and only if

$$\int_0^1 dr\ f(r)\ \ln\left(\frac{1}{r}\right) < \infty.$$

-Iif $d > 2$ or $\beta \in]0, \frac{1}{2}[$, then (7.30) is satisfied if and only if

$$\int_0^1 dr\ f(r)\ \frac{1}{r^{2\beta}} < \infty.$$

Finally, the reformulation of condition B_0', though not established because of technical difficulties, is conjectured to give the following.

- If $d = 2$, we expect that Assumption B_0' is satisfied (perhaps modulo a boundedness assumption) if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1)\ \ln\left(\frac{1}{|x_1|}\right)^2 < \infty.$$

- If $d > 2$, we expect that Assumption B_0' is satisfied (perhaps modulo a boundedness assumption) if and only if

$$\int_{B_1(0,1)} \Gamma(dx_1)\ \frac{1}{|x_1|^{d-2}}\ \ln\left(\frac{1}{|x_1|}\right) < \infty.$$

7.6 Noise on a lower dimensional plane

Let us first consider the case where the noise is concentrated on a k-plane, with $k = d-2$. Using (4.1), (7.1) and [45, formulas I.14.16, I.14.55 and I.18.31], we can compute the Fourier transform of G in the first $d-2$ coordinates of x. For $(t, \xi_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-2} \times \mathbb{R}^2$,

$$\mathcal{F}_1 G(t,\xi_1,x_2) = \begin{cases} \dfrac{e^{-at}}{2\pi}\ \dfrac{\cos\left(\sqrt{(|\xi_1|^2 + b - a^2)\ (t^2 - |x_2|^2)}\right)}{\sqrt{t^2 - |x_2|^2}}\ 1_{\{|x_2| < t\}}, & \text{if } \ |\xi_1|^2 \geq a^2 - b, \\[2ex] \dfrac{e^{-at}}{2\pi}\ \dfrac{\cosh\left(\sqrt{(a^2 - b - |\xi_1|^2)\ (t^2 - |x_2|^2)}\right)}{\sqrt{t^2 - |x_2|^2}}\ 1_{\{|x_2| < t\}}, & \text{if } \begin{cases} a^2 - b > 0 \text{ and} \\ |\xi_1|^2 < a^2 - b. \end{cases} \end{cases} \tag{7.31}$$

The next theorem shows that when $k = d-2$, the distribution-valued solution u of equation (6.5) cannot be a real-valued process, even outside the k-plane $x_2 = 0$.

Theorem 7.6.1. *Let u be the solution of equation (6.5) with $k = d-2$. Then there does not exist a real-valued square integrable process $X = \{X(t,x_1,x_2),\ (t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k} \backslash \{0\}\}$*

such that the map $(t, x_1, x_2) \mapsto X(t, x_1, x_2)$ *is continuous from* $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}$ *to* $L^2(\Omega)$ *and*

$$\langle u(t), \varphi \rangle = \int_{\mathbb{R}^d} dx\, X(t,x)\, \varphi(x), \qquad \mathbb{P} - a.s.,$$

for all $t \in \mathbb{R}_+$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *such that supp* $\varphi \subset \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}$.

Proof. Suppose that there exists a process X satisfying the above conditions. As in the proof of Theorem 7.2.5, fix $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-2} \times \mathbb{R}^2$ such that $0 < |x_2| < t$ and let $\varphi^{(n)}_{x_1,x_2} = \delta_{(x_1,x_2)} * \psi_n \in \mathcal{S}(\mathbb{R}^d)$, where (ψ_n) is a sequence of non-negative and compactly supported approximations of δ_0 in $\mathbb{R}^d$. By the assumptions made on X, we obtain that

$$\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,x_2}\rangle|^2) = \mathbb{E}(X(t,x_1,x_2)^2) < \infty, \tag{7.32}$$

and also that

$$\begin{aligned}\mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,x_2}\rangle|^2) &= \int_{\mathbb{R}^{d-2}} \mu(d\xi_1) \int_0^t ds\, \left|\mathcal{F}_1(G(t-s) * \varphi^{(n)}_{x_1,x_2})(\xi_1, 0)\right|^2 \\ &= \int_{\mathbb{R}^{d-2}} \mu(d\xi_1) \int_0^t ds\, \left|\mathcal{F}_1(G(s) * \varphi^{(n)}_{x_1,x_2})(\xi_1, 0)\right|^2.\end{aligned}$$

Moreover, by arguments similar to those used in the proof of Theorem 7.2.5, we obtain that for all $s \neq |x_2|$ and $\xi_1 \in \mathbb{R}^{d-2}$,

$$\mathcal{F}_1(G(s) * \varphi^{(n)}_{x_1,x_2})(\xi_1, 0) \underset{n\to\infty}{\rightarrow} \mathcal{F}_1 G(s, \xi_1, -x_2)\, \chi_{x_1}(\xi_1),$$

where $\mathcal{F}_1 G$ is given by (7.31). So Fatou's lemma tells us that

$$\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,x_2}\rangle|^2) \geq \int_{\mathbb{R}^{d-2}} \mu(d\xi_1) \int_{|x_2|}^t ds\, \mathcal{F}_1 G(s, \xi_1, -x_2)^2. \tag{7.33}$$

For a fixed $\xi_1 \in \mathbb{R}^{d-2}$ such that $|\xi_1|^2 \geq a^2 - b$:

$$\int_{|x_2|}^t ds\, \mathcal{F}_1 G(s, \xi_1, -x_2)^2 \geq \frac{e^{-2at}}{4\pi^2} \int_{|x_2|}^t ds\, \frac{\cos\left(\sqrt{(s^2 - |x_2|^2)\,(|\xi_1|^2 + b - a^2)}\right)^2}{s^2 - |x_2|^2} = \infty,$$

because for $\varepsilon \in\,]0,1[$ fixed, there exists $\delta > 0$ such that

$$\cos^2\left(\sqrt{(s^2 - |x_2|^2)\,(|\xi_1|^2 + b - a^2)}\right) \geq 1 - \varepsilon, \qquad \forall s \in [|x_2|, |x_2| + \delta],$$

and

$$\int_{|x_2|}^{|x_2|+\delta} ds\, \frac{1-\varepsilon}{s^2 - |x_2|^2} = \infty.$$

The right-hand side of (7.33) is therefore infinite, but this is a contradiction since the limit on the left-hand side of (7.33) exists and is finite by (7.32). We thus have proven that a process X satisfying all the conditions of the theorem cannot exist. □

When $k < d-2$, $\mathcal{F}_1 G$ is no longer a function, but a distribution in ξ_1. We therefore expect that the following limit

$$\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,x_2}\rangle|^2) = \int_{\mathbb{R}^{d-k}} \mu(d\xi_1) \int_0^t ds \left|\mathcal{F}_1(G(t-s) * \varphi^{(n)}_{x_1,x_2})(\xi_1, 0)\right|^2$$

is also infinite in this case, independently of any condition on μ, so that there never exists a real-valued solution. Nevertheless, this fact has not been proven analytically.

We will show in Chapter 9 that the situation is once again completely different in the case of the heat equation, for which there always exists a real-valued solution outside the k-plane $x_2 = 0$ for any $k \in \{1, ..., d-1\}$.

Chapter 8

Non-linear hyperbolic equation in $\mathbb{R}^d$ driven by noise on a hyperplane

In the preceding chapter, we have obtained precise conditions which guarantee the existence of a real-valued weak solution u of the linear hyperbolic equation (6.2) in the case of a noise concentrated on the hyperplane $x_2 = 0$, and also some regularity of this solution. With these results in hand, we are now able to treat non-linear equations of the same type.

For technical reasons, we make the following assumption.

Assumption C_0.
(i) The covariance Γ of the noise is a non-negative measure on $\mathbb{R}^{d-1}$.

(ii) $d \in \{2, 3\}$ (so the "hyperplane" $x_2 = 0$ becomes either a straight line or a plane) and $a^2 \geq b$.

From the expressions of G listed in Appendix A, we see that part (ii) of this assumption implies that for all $t \in \mathbb{R}_+$ and $x_2 \in \mathbb{R}$, the Green kernel $G(t, \cdot, x_2)$ is a non-negative measure on $\mathbb{R}^{d-1}$. These non-negativity assumptions are necessary in Theorem 6.2.1, which we shall use repeatedly in the following.

8.1 Non-linear term restricted to the hyperplane

Let us consider the following formal non-linear equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) + b\,u(t,x) - \Delta u(t,x) \\ \qquad = g(u(t,x_1,0))\ \delta_0(x_2) + h(u(t,x_1,0))\ \dot{F}(t,x_1)\ \delta_0(x_2), \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\ u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0, \qquad x \in \mathbb{R}^d, \end{cases} \tag{8.1}$$

where g and h are real-valued functions and $\dot{F}$ is the noise concentrated on the hyperplane $x_2 = 0$ considered in Chapter 6 (with $k = d - 1$). Note that we consider null initial conditions, but this could be improved; see Remark 8.1.3.

The non-linear term in this equation is restricted to the hyperplane $x_2 = 0$ and composed by a deterministic part $g(u)$ and a stochastic one $h(u)\dot{F}$. If we consider that it is a given function of (t, x), we can then write formally what should be the solution of this "linear" equation, using the extended definition of the stochastic integral of Section 6.2:

$$\begin{aligned} u(t, x_1, x_2) &= \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s, dz_1, x_2)\, g(u(t-s, x_1 - z_1, 0)) \\ &\quad + \int_{[0,t]\times\mathbb{R}^{d-1}} M(ds, dz_1)\, h(u(s, z_1, 0))\, G(t-s, x_1 - z_1, x_2), \end{aligned} \tag{8.2}$$

$\mathbb{P}$-a.s., for all $(t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}$, where G is the solution of equation (5.15) and M is the worthy martingale measure defined in Section 6.1. Actually, since the non-linear term contains the unknown u, what we have obtained here is a rigorous formulation of equation (8.1); a *mild solution* of equation (8.1) is a predictable process $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ which satisfies the above equation.

Note that when $g \equiv 0$ and $h \equiv 1$, the solution of the above "equation" is precisely the real-valued solution of equation (6.5), which is well defined on the whole space (see Theorem 7.3.3) when Assumption B_0' is satisfied, namely when

$$\int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)\ \ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}} < \infty.$$

The following theorem states that under this assumption, there still exists a real-valued solution to equation (8.2) when g and h are globally Lipschitz functions.

Theorem 8.1.1. *Under Assumptions B_0' and C_0, and if g and h are globally Lipschitz functions, then there exists a unique mild solution $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ to equation (8.1). Moreover, the map $(t,x) \mapsto u(t,x)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^d$ to $L^2(\Omega)$ and for all $T > 0$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(u(t,x)^2) < \infty. \tag{8.3}$$

Before proving this theorem, let us mention that the Green kernel restricted to the hyperplane $x_2 = 0$ plays a crucial role here; in particular, we will need the following lemma on the behavior of $\mathcal{F}_1 G$ evaluated in $x_2 = 0$.

Lemma 8.1.2. *For all $t > 0$, there exists $C_{11}(t) > 0$ such that*

$$|\mathcal{F}_1 G(s, \xi_1, 0)| \le \frac{C_{11}(t)}{(1 + (|\xi_1| s)^2)^{\frac{1}{4}}}, \qquad \forall (s, \xi_1) \in [0,t] \times \mathbb{R}^{d-1}.$$

Proof of Lemma 8.1.2. If $|\xi_1|^2 \ge 2\,(a^2 - b)$, then since

$$|J_0(r)| \le \frac{C}{(1+r^2)^{\frac{1}{4}}}, \qquad \forall r \ge 0,$$

we obtain by (7.1) that

$$|\mathcal{F}_1 G(s,\xi_1,0)| = \frac{e^{-as}}{2}\,\left|J_0\left(\sqrt{(|\xi_1|^2+b-a^2)}\;s\right)\right| \le \frac{e^{a^-t}}{2}\,\frac{C}{(1+(|\xi_1|^2+b-a^2)\;s^2)^{\frac14}}.$$

But $|\xi_1|^2+b-a^2 \ge \frac{|\xi_1|^2}{2}$, so

$$|\mathcal{F}_1 G(s,\xi_1,0)| \le C\,\frac{e^{a^-t}}{2}\left(1+\frac{(|\xi_1|s)^2}{2}\right)^{-\frac14} \le C\,e^{a^-t}\,\frac{1}{(1+(|\xi_1|s)^2)^{\frac14}}.$$

If now $a^2-b \le |\xi_1|^2 \le 2\,(a^2-b)$, then since $|J_0(r)| \le 1$ for all $r \ge 0$,

$$|\mathcal{F}_1 G(s,\xi_1,0)| = \frac{e^{-as}}{2}\,\left|J_0\left(\sqrt{(|\xi_1|^2+b-a^2)}\;s\right)\right| \le \frac{e^{a^-t}}{2}.$$

Finally, if $|\xi_1|^2 \le a^2-b$, then since $|I_0(r)| \le C\,e^r$ for all $r \ge 0$,

$$\begin{aligned}|\mathcal{F}_1 G(s,\xi_1,0)| &= \frac{e^{-as}}{2}\,\left|I_0\left(\sqrt{(a^2-b-|\xi_1|^2)}\;s\right)\right|\\ &\le \frac{e^{a^-t}}{2}\,C\,e^{\sqrt{a^2-b-|\xi_1|^2}\,s} \le \frac{e^{a^-t}}{2}\,C\,e^{\sqrt{a^2-b}\,t},\end{aligned}$$

and the proof ends as the proof of Lemma 5.4.1. □

With this tool in hand, we can now prove the theorem.

Proof of Theorem 8.1.1. Let us consider $v(t,x_1) = u(t,x_1,0)$, $(t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$. Equation (8.2) evaluated in $x_2 = 0$ gives the following (closed) equation for v:

$$\begin{aligned}v(t,x_1) &= \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,0)\;g(v(t-s,x_1-z_1))\\ &\quad+\int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dz_1)\;v(s,z_1)\;G(t-s,x_1-z_1,0).\end{aligned} \tag{8.4}$$

Although $G(\cdot,\cdot,0)$ is not the Green kernel of any "standard" equation in $\mathbb{R}_+ \times \mathbb{R}^{d-1}$, the above equation is of the type of the ones studied in [15]. We can therefore apply Theorem 13 of that paper; in order to do this, we need to verify (see [16]) that for all $t \in \mathbb{R}_+$, $G(t,\cdot,0) \in O'_C(\mathbb{R}^{d-1})_+$ (which is clear from the definition of G and the assumptions made on d, a and b), that for all $\xi_1 \in \mathbb{R}^{d-1}$, the map $t \mapsto \mathcal{F}_1 G(t,\xi_1,0)$ is continuous (which is also clear from the expression of $\mathcal{F}_1 G(t,\xi_1,0)$ in (7.1)) and finally that for all $t > 0$, there exists $h_0 > 0$ and $k : [0,t] \to O'_C(\mathbb{R}^{d-1})_+$ such that for all $s \in [0,t]$, $h \in [0,h_0]$ and $\xi_1 \in \mathbb{R}^{d-1}$,

$$|\mathcal{F}_1 G(s+h,\xi_1,0) - \mathcal{F}_1 G(s,\xi_1,0)| \le \mathcal{F}_1 k(s,\xi_1), \tag{8.5}$$

and

$$\int_0^t ds \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\;\mathcal{F}_1 k(s,\xi_1)^2 < \infty. \tag{8.6}$$

By Lemma 8.1.2, the distribution-valued function k whose Fourier transform is defined by

$$\mathcal{F}_1 k(s,\xi_1) = \frac{2\,C_{11}(t)}{(1+(|\xi_1|s)^2)^{\frac14}},$$

satifies (8.5). Note also that for all $s \in [0,t]$, $\mathcal{F}_1 k(s,\cdot) \in O_M(\mathbb{R}^{d-1})$, so $k(s,\cdot) \in O'_C(\mathbb{R}^{d-1})$ by (4.2); moreover, $k(s,\cdot)$ is a non-negative distribution on $\mathbb{R}^{d-1}$ since when $s=0$, $k(s,\cdot) = 2\,C_{11}(t)\,\delta_0(\cdot)$ which is non-negative, and when $s>0$, we have by [45, formulas I.2.7 and I.18.29]:

$$k(s,x_1) = \frac{\tilde{C}\,C_{11}(t)}{\sqrt{s}\,|x_1|^{\frac{1}{4}}}\,K_{\frac{1}{4}}\left(\frac{|x_1|}{s}\right), \qquad \text{when } d=2,$$

and

$$k(s,x_1) = \frac{\hat{C}\,C_{11}(t)}{s\,|x_1|^{\frac{3}{4}}}\,K_{\frac{3}{4}}\left(\frac{|x_1|}{s}\right), \qquad \text{when } d=3,$$

where K_ν is the modified Bessel function of order ν of the second kind, which is non-negative on $\mathbb{R}_+$ (see Appendix B). By estimates analogous to those in the proof of Lemma 7.3.1, we also obtain that

$$\begin{aligned}\int_0^t ds \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\,\mathcal{F}_1 k(s,\xi_1)^2 &= \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\,\frac{4\,C_{11}(t)^2}{\sqrt{1+(|\xi_1|s)^2}} \\ &\leq 4\,C_{11}(t)^2\,C_7(t) \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\,\frac{1+\ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}},\end{aligned}$$

which is finite by Assumption B'_0, so (8.6) is proven. Theorem 13 of [15] then states that there exists a unique predictable process v which satisfies (8.4). Moreover, the distribution of $v(t,x_1)$ is stationnary in x_1, the map $(t,x_1) \mapsto v(t,x_1)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^{d-1}$ to $L^2(\Omega)$ (note that the continuity in t is uniform in $x_1 \in \mathbb{R}^{d-1}$) and for all $T>0$,

$$\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \mathbb{E}(v(t,x_1)^2) < \infty. \tag{8.7}$$

So $u(t,x_1,0) = v(t,x_1)$ gives the solution of equation (8.2) on the hyperplane $x_2 = 0$. For $x_2 \neq 0$, let us now *define* $u(t,x_1,x_2)$ by

$$u(t,x_1,x_2) = \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\,g(v(t-s,x_1-z_1)) + (G(t-\cdot,x_1-\cdot,x_2)\cdot M^{h(v)})_t,$$

which is not anymore an equation, since v is now a given process (note that since $G(t-\cdot,x_1-\cdot,x_2)$ is non-negative, $\|G(t-\cdot,x_1-\cdot,x_2)\|_t < \infty$ and $Z = h(v)$ satisfies conditions (6.8) and (6.10), the stochastic integral is well defined by Theorem 6.2.1). This shows directly that u satisfies equation (8.2) and moreover that it is the unique process to do so. Moreover, it admits a jointly measurable modification since it is continuous in $L^2(\Omega)$, what we now prove. To this end, write

$$u(t,x_1,x_2) = A(t,x_1,x_2) + B(t,x_1,x_2),$$

where

$$A(t,x_1,x_2) = \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\,g(v(t-s,x_1-z_1))$$

and

$$B(t,x_1,x_2) = (G(t-\cdot,x_1-\cdot,x_2)\cdot M^{h(v)})_t.$$

We first verify the L^2-continuity of the process B, following the scheme of the proof of Lemma 7.2.2 and using the precise estimates of Theorem 7.3.3. Let us therefore consider $x_2,\ y_2 \in \mathbb{R}$;

$$\begin{aligned}
&\sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \mathbb{E}((B(t,x_1,y_2) - B(t,x_1,x_2))^2) \\
&= \sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu_s^{h(v)}(d\xi_1)\ (\mathcal{F}_1 G(t-s,-\xi_1,y_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2 \\
&\leq \sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \int_0^t ds \sup_{z_1\in\mathbb{R}^{d-1}} \mathbb{E}(h(v(s,z_1))^2) \\
&\qquad\qquad \cdot \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\ (\mathcal{F}_1 G(t-s,-\xi_1,y_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2,
\end{aligned}$$

by Theorem 6.2.1. Using the global Lipschitz property of h (which implies linear growth), we find that this expression is less than or equal to

$$\begin{aligned}
&\sup_{(s,z_1)\in[0,T]\times\mathbb{R}^{d-1}} K^2\ \mathbb{E}(1 + v(s,z_1)^2) \cdot \\
&\quad \sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ (\mathcal{F}_1 G(t-s,-\xi_1,y_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2,
\end{aligned}$$

which converges to 0 as $y_2 \to x_2$, by (8.7) and the same argument as in proof of Theorem 7.3.3.

For $x_1,\ y_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}((B(t,y_1,x_2) - B(t,x_1,x_2))^2) \\
&= \sup_{t\in[0,T]} \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu_s^{h(v)}(d\xi_1)\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2\ 2\ (1 - \cos(\xi_1 \cdot (y_1 - x_1))) \\
&\leq \sup_{t\in[0,T]} \int_0^t ds \sup_{z_1\in\mathbb{R}^{d-1}} \mathbb{E}(h(v(s,z_1))^2) \\
&\qquad \cdot \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2\ 2\ (1 - \cos(\xi_1 \cdot (y_1 - x_1))) \\
&\leq \sup_{(s,z_1)\in[0,T]\times\mathbb{R}^{d-1}} K^2\ \mathbb{E}(1 + v(s,z_1)^2) \\
&\qquad \cdot \sup_{t\in[0,T]} \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2\ 2\ (1 - \cos(\xi_1 \cdot (y_1 - x_1))),
\end{aligned}$$

which converges to 0 as $y_1 \to x_1$.

Considering finally $t, h \in \mathbb{R}_+$, $x_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\mathbb{E}((B(t+h,x_1,x_2) - B(t,x_1,x_2))^2) \\
&\quad = \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu_s^{h(v)}(d\xi_1)\ (\mathcal{F}_1 G(t+h-s,-\xi_1,x_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2 \\
&\qquad + \int_t^{t+h} ds \int_{\mathbb{R}^{d-1}} \mu_s^{h(v)}(d\xi_1)\ \mathcal{F}_1 G(t+h-s,-\xi_1,x_2)^2 \\
&\quad \leq \sup_{(s,z_1)\in[0,t]\times\mathbb{R}^{d-1}} K^2\ \mathbb{E}(1+v(s,z_1)^2) \\
&\qquad\qquad \cdot \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ (\mathcal{F}_1 G(t+h-s,-\xi_1,x_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2 \\
&\qquad + \sup_{(s,z_1)\in[t,t+h_0]\times\mathbb{R}^{d-1}} K^2\ \mathbb{E}(1+v(s,z_1)^2) \\
&\qquad\qquad \cdot \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_t^{t+h} ds\ \mathcal{F}_1 G(t+h-s,-\xi_1,x_2)^2,
\end{aligned}$$

for all $h \leq h_0$, and this expression converges to 0 as $h \to 0$. Since a similar estimate holds for $h \leq 0$, we have shown the L^2-continuity of the process B.

We now prove the L^2-continuity of the process A following a different order: we first show that the map $t \mapsto A(t,x)$ is continuous in $L^2(\Omega)$ uniformly in $x \in \mathbb{R}^d$, then that for fixed $t \in \mathbb{R}_+$, the map $x_1 \mapsto A(t,x_1,x_2)$ is continuous in $L^2(\Omega)$ uniformly in $x_2 \in \mathbb{R}$ and finally that for fixed $(t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$, the map $x_2 \mapsto A(t,x_1,x_2)$ is continuous in $L^2(\Omega)$.

Let therefore $t, h \in \mathbb{R}_+$;

$$\begin{aligned}
&\sup_{(x_1,x_2)\in\mathbb{R}^{d-1}\times\mathbb{R}} \mathbb{E}((A(t+h,x_1,x_2) - A(t,x_1,x_2))^2) \\
&\quad = \sup_{(x_1,x_2)\in\mathbb{R}^{d-1}\times\mathbb{R}} \mathbb{E}\Bigg(\bigg(\int_0^{t+h} ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ g(v(t+h-s,x_1-z_1)) \\
&\qquad\qquad - \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ g(v(t-s,x_1-z_1))\bigg)^2\Bigg) \\
&\quad \leq 2 \sup_{(x_1,x_2)\in\mathbb{R}^{d-1}\times\mathbb{R}} \mathbb{E}\Bigg(\bigg(\int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2) \\
&\qquad\qquad \cdot (g(v(t+h-s,x_1-z_1)) - g(v(t-s,x_1-z_1)))\bigg)^2\Bigg) \\
&\qquad + 2 \sup_{(x_1,x_2)\in\mathbb{R}^{d-1}\times\mathbb{R}} \mathbb{E}\Bigg(\bigg(\int_t^{t+h} ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ g(v(t+h-s,x_1-z_1))\bigg)^2\Bigg).
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the Lipschitz property of g, we find that this last expression is less than or equal to

$$\begin{aligned}
& 2 \sup_{(x_1,x_2)\in\mathbb{R}^{d-1}\times\mathbb{R}} \int_0^t ds\, G(s,\mathbb{R}^{d-1},x_2) \\
& \qquad \cdot \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ K^2\ \mathbb{E}((v(t+h-s,x_1-z_1)-v(t-s,x_1-z_1))^2) \\
+ \ & 2 \sup_{(x_1,x_2)\in\mathbb{R}^{d-1}\times\mathbb{R}} \int_t^{t+h} ds\, G(s,\mathbb{R}^{d-1},x_2) \\
& \qquad \cdot \int_t^{t+h} ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ K^2\ \mathbb{E}(1+v(t+h-s,x_1-z_1)^2).
\end{aligned}$$

Note that

$$\begin{aligned}
G(s,\mathbb{R}^{d-1},x_2) &= \mathcal{F}_1 G(s,0,x_2) = \frac{e^{-as}}{2}\, I_0\left(\sqrt{a^2-b}\,\sqrt{s^2-x_2^2}\right)\, 1_{\{|x_2|\,<\,s\}} \\
&\leq \frac{e^{a^- s}}{2}\, C\, e^{\sqrt{a^2-b}\, s} = C_s, \qquad (8.8)
\end{aligned}$$

since $I_0(r) \leq C\, e^r$ for all $r \geq 0$, by (B.3). So after introducing the supremum over x_1 under the integral sign in the above expression, we obtain that it is less than or equal to

$$\begin{aligned}
& 2\, C_t\, K^2\, t \sup_{x_2\in\mathbb{R}} \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2) \sup_{x_1\in\mathbb{R}^{d-1}} \mathbb{E}((v(t+h-s,x_1)-v(t-s,x_1))^2) \\
& + 2\, C_{t+h_0}\, K^2\, h \sup_{x_2\in\mathbb{R}} \int_t^{t+h} ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2) \sup_{x_1\in\mathbb{R}^{d-1}} \mathbb{E}(1+v(t+h-s,x_1)^2) \\
\leq\ & 2\, C_t^2\, K^2\, t \int_0^t ds \sup_{x_1\in\mathbb{R}^{d-1}} \mathbb{E}((v(t+h-s,x_1)-v(t-s,x_1))^2) \\
& + 2\, C_{t+h_0}^2\, K^2\, h^2 \sup_{(s,x_1)\in[0,h_0]\times\mathbb{R}^{d-1}} \mathbb{E}(1+v(s,x_1)^2),
\end{aligned}$$

for all $h \leq h_0$. By the dominated convergence theorem, the first term of this expression converges to 0 as $h \to 0$, since the process v is L^2-continuous in t, uniformly in $x_1 \in \mathbb{R}^{d-1}$ (see [15] and [16]) and for all $h \leq h_0$,

$$\sup_{(s,x_1)\in[0,t]\times\mathbb{R}^{d-1}} \mathbb{E}((v(t+h-s,x_1)-v(t-s,x_1))^2) \leq 2 \sup_{(s,x_1)\in[0,t+h_0]\times\mathbb{R}^{d-1}} \mathbb{E}(v(s,x_1)^2) < \infty,$$

by (8.7). The same conclusion is immediate for the second term: similar estimates give the convergence to 0 for $h \leq 0$.

Let now $t \in \mathbb{R}_+$ and x_1, $y_1 \in \mathbb{R}^{d-1}$;

$$\begin{aligned}
& \sup_{x_2\in\mathbb{R}} \mathbb{E}((A(t,y_1,x_2)-A(t,x_1,x_2))^2) \\
=\ & \sup_{x_2\in\mathbb{R}} \mathbb{E}\left(\left(\int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ (g(v(t-s,y_1-z_1))-g(v(t-s,x_1-z_1)))\right)^2\right).
\end{aligned}$$

Using, as before, the Cauchy-Schwarz inequality and the Lipschitz property of g, we find that this last expression is less than or equal to

$$\sup_{x_2\in\mathbb{R}} \int_0^t ds\, G(s,\mathbb{R}^{d-1},x_2)$$
$$\cdot \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\; K^2\; \mathbb{E}((v(t-s,y_1-z_1)-v(t-s,x_1-z_1))^2).$$

By (8.8) and the stationnarity of the distribution of v in x_1, this expression is less than or equal to

$$C_t\; K^2\; t \sup_{x_2\in\mathbb{R}} \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\; \mathbb{E}((v(t-s,y_1)-v(t-s,x_1))^2)$$
$$\leq\; C_t^2\; K^2\; t \int_0^t ds\; \mathbb{E}((v(t-s,y_1)-v(t-s,x_1))^2).$$

Now, since the process v is L^2-continuous in x_1 and satisfies (8.7), we conclude by the dominated convergence theorem that the above expression converges to 0 as $y_1 \to x_1$.

Finally, let $t\in\mathbb{R}_+$, $x_1\in\mathbb{R}^{d-1}$ and x_2, $y_2\in\mathbb{R}$;

$$\mathbb{E}((A(t,x_1,y_2)-A(t,x_1,x_2))^2)$$
$$= \mathbb{E}\Bigg(\Big(\int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,y_2)\; g(v(t-s,x_1-z_1))$$
$$- \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\; g(v(t-s,x_1-z_1))\Big)^2\Bigg).$$

At this point, we need to consider separately the two cases $d=2$ and $d=3$. Let us begin by the case $d=2$: using (A.12) and the change of variable $z_1=\sqrt{s^2-x_2^2}\; w_1$, we obtain for a measurable function h defined on $\mathbb{R}$,

$$\int_{\mathbb{R}} G(s,dz_1,x_2)\; h(z_1)$$
$$= 1_{\{|x_2|<s\}}\; \frac{e^{-as}}{2\pi} \int_{|z_1|<\sqrt{s^2-x_2^2}} dz_1\; \frac{\cosh\left(\sqrt{(a^2-b)\;(s^2-x_2^2-z_1^2)}\right)}{\sqrt{s^2-x_2^2-z_1^2}}\; h(z_1)$$
$$= 1_{\{|x_2|<s\}}\; \frac{e^{-as}}{2\pi} \int_{|w_1|<1} dw_1\; \frac{\cosh\left(\sqrt{(a^2-b)\;(s^2-x_2^2)}\;|w_1|\right)}{\sqrt{1-w_1^2}}\; h\left(\sqrt{s^2-x_2^2}\; w_1\right).$$

So

$$\int_{\mathbb{R}} G(s,dz_1,y_2)\; h(z_1) - \int_{\mathbb{R}} G(s,dz_1,x_2)\; h(z_1)$$
$$= \frac{e^{-as}}{2\pi} \int_{|w_1|<1} \frac{dw_1}{\sqrt{1-|w_1|^2}}\; (H(s,w_1,y_2)-H(s,w_1,x_2)).$$

where

$$H(s,w_1,x_2) = \cosh\left(\sqrt{(a^2-b)\;(s^2-x_2^2)}\;|w_1|\right)\; h\left(\sqrt{s^2-x_2^2}\; w_1\right)\; 1_{\{|x_2|<s\}}.$$

Using the above equality with $h(z_1) = g(v(t-s, x_1 - z_1))$ and the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}((A(t,x_1,y_2) - A(t,x_1,x_2))^2) \leq \int_0^t ds\, \frac{e^{-as}}{2\pi} \int_{|w_1|<1} \frac{dw_1}{\sqrt{1-|w_1|^2}}$$

$$\cdot \int_0^t ds\, \frac{e^{-as}}{2\pi} \int_{|w_1|<1} \frac{dw_1}{\sqrt{1-|w_1|^2}}\, \mathbb{E}((H(s,w_1,y_2) - H(s,w_1,x_2))^2).$$

Since

$$\mathbb{E}((H(s,w_1,y_2) - H(s,w_1,x_2))^2) \underset{y_2 \to x_2}{\to} 0,$$

for $s \neq |x_2|$ and $w_1 \in B_1(0,1)$, by the continuity of cosh, the Lipschitz property of g and the L^2-continuity of v; since moreover H is bounded by the Lipschitz property of g and (8.7), and since finally

$$\int_0^t ds\, \frac{e^{-as}}{2\pi} \int_{|w_1|<1} \frac{dw_1}{\sqrt{1-|w_1|^2}} < \infty,$$

we obtain by the dominated convergence theorem that when $d = 2$,

$$\mathbb{E}((A(t,x_1,y_2) - A(t,x_1,x_2))^2) \underset{y_2 \to x_2}{\to} 0.$$

When $d = 3$, using (A.13) and the change of variable $z_1 = \sqrt{s^2 - x_2^2}\; w_1$, we obtain for a measurable function h defined on $\mathbb{R}^2$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} G(s,dz_1,x_2)\, h(z_1) \\
&= 1_{\{|x_2| < s\}} \frac{e^{-as}}{4\pi} \Bigg(\frac{1}{\sqrt{s^2 - x_2^2}} \int_{|z_1| = \sqrt{s^2-x_2^2}} d\tau(z_1)\, h(z_1) \\
&\quad + \sqrt{a^2 - b} \int_{|z_1| < \sqrt{s^2 - x_2^2}} dz_1\, \frac{I_1\left(\sqrt{(a^2-b)\,(s^2 - x_2^2 - |z_1|^2)}\right)}{\sqrt{s^2 - x_2^2 - |z_1|^2}}\, h(z_1) \Bigg) \\
&= 1_{\{|x_2| < s\}} \frac{e^{-as}}{4\pi} \Bigg(\int_{|w_1|=1} d\tau(w_1)\, h\left(\sqrt{s^2 - x_2^2}\, w_1\right) \\
&\quad + \sqrt{(a^2-b)\,(s^2-x_2^2)} \int_{|w_1|<1} dw_1\, \frac{I_1\left(\sqrt{(a^2-b)\,(s^2-x_2^2)}\,|w_1|\right)}{\sqrt{1-|w_1|^2}}\, h\left(\sqrt{s^2-x_2^2}\, w_1\right) \Bigg),
\end{aligned}$$

where I_1 is the first order modified Bessel function of the first kind (see Appendix B). So

$$\begin{aligned}
&\int_{\mathbb{R}^2} G(s,dz_1,y_2)\, h(z_1) - \int_{\mathbb{R}^2} G(s,dz_1,x_2)\, h(z_1) \\
&= \frac{e^{-as}}{4\pi} \Bigg(\int_{|w_1|=1} d\tau(w_1)\, (\tilde{H}(s,w_1,y_2) - \tilde{H}(s,w_1,x_2)) \\
&\quad + \int_{|w_1|<1} \frac{dw_1}{\sqrt{1-|w_1|^2}}\, (\hat{H}(s,w_1,y_2) - \hat{H}(s,w_1,x_2)) \Bigg),
\end{aligned}$$

where

$$\tilde{H}(s,w_1,x_2) = h\left(\sqrt{s^2-x_2^2}\, w_1\right)\, 1_{\{|x_2| < s\}}$$

and

$$\hat{H}(s, w_1, x_2) = I_1\left(\sqrt{(a^2-b)\,(s^2-x_2^2)}\,|w_1|\right)\ h\left(\sqrt{s^2-x_2^2}\,w_1\right)\ 1_{\{|x_2|\,<\,s\}}.$$

Applying the same technique as before, we conclude that

$$\mathbb{E}((A(t,x_1,y_2)-A(t,x_1,x_2))^2) \underset{y_2\to x_2}{\to} 0$$

also in the case $d=3$. So we have shown the L^2-continuity of the processes A and B, therefore that of the process u. Let us now check that u satisfies (8.3), verifying separately that both processes A and B do:

$$\begin{aligned}
\mathbb{E}(A(t,x_1,x_2)^2) &= \mathbb{E}\left(\left(\int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ g(v(t-s,x_1-z_1))\right)^2\right)\\
&\leq \int_0^t ds\ G(s,\mathbb{R}^{d-1},x_2)\\
&\qquad\quad \cdot \int_0^t ds \int_{\mathbb{R}^{d-1}} G(s,dz_1,x_2)\ K^2\ \mathbb{E}(1+v(t-s,x_1-z_1)^2),
\end{aligned}$$

by the Cauchy-Schwarz inequality and the global Lipschitz property of g. By estimates (8.7) and (8.8), this expression is less than or equal to

$$C_t^2\ K^2\ t^2 \sup_{(s,x_1)\in[0,t]\times\mathbb{R}^{d-1}} \mathbb{E}(1+v(s,x_1)^2),$$

so

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(A(t,x)^2) \leq C_T^2\ K^2\ T^2 \sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \mathbb{E}(1+v(t,x_1)^2) < \infty.$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E}(B(t,x_1,x_2)^2) &= \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu_s^{h(v)}(d\xi_1)\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2\\
&\leq \int_0^t ds \sup_{y_1\in\mathbb{R}^{d-1}} \mathbb{E}(1+v(s,y_1)^2) \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2,
\end{aligned}$$

by Theorem 6.2.1 and the global Lipschitz property of h. But this last expression is also less than or equal to

$$\sup_{(s,x_1)\in[0,t]\times\mathbb{R}^{d-1}} \mathbb{E}(1+v(s,x_1)^2) \int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^t ds\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2.$$

So finally,

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(B(t,x)^2)\\
&\leq \sup_{(t,x_1)\in[0,T]\times\mathbb{R}^{d-1}} \mathbb{E}(1+v(t,x_1)^2)\ C_7(T) \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\ \frac{1+\ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}} < \infty,
\end{aligned}$$

by Lemma 7.3.1 and Assumption B_0'. The process u satifies then (8.3) and we can apply Proposition 2 of [14] to conclude that u is predictable. This completes the proof. □

Remark 8.1.3. From the preceding proof, we see that the solution $u(t, x_1, 0)$ of the equation restricted to the hyperplane $x_2 = 0$ is spatially homogeneous in x_1 (that is, its statistical properties do not depend on x_1). Moreover, it turns out also from the preceding proof that if we want to add initial conditions to the equation, we need to assume that they do not depend on x_1 if we want to leave the proof unchanged.

8.2 Global semi-linear equation

We study here the following formal semi-linear equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) - \Delta u(t,x) \\ \qquad = g(u(t,x)) + \dot{F}(t,x_1)\,\delta_0(x_2), \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\ u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0, \qquad x \in \mathbb{R}^d, \end{cases} \tag{8.9}$$

where g is a real-valued function and $\dot{F}$ is the noise concentrated on the hyperplane $x_2 = 0$ considered in Chapter 6 (with $k = d - 1$).

The solution of this equation represents a wave generated by a non-linear source g distributed on the whole space with an additive noise term on the hyperplane $x_2 = 0$.

Note that the coefficient b of equation (6.5) is equal here to 0. The reason for this is that the term bu can be included in the term $g(u)$. So part (ii) of Assumption C_0 made at the beginning of this chapter only imposes here that $d \in \{2, 3\}$.

Following the argument of the preceding section, a mild solution of equation (8.9) is then a predictable process $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ which satisfies

$$\begin{aligned} u(t,x) &= \int_0^t ds \int_{\mathbb{R}^d} G(s,dz)\, g(u(t-s, x-z)) \\ &\quad + \int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dz_1)\, G(t-s, x_1 - z_1, x_2), \end{aligned} \tag{8.10}$$

$\mathbb{P}$-a.s., for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where G is the solution of equation (5.15) with parameter $b = 0$.

We can now state the following existence and uniqueness theorem.

Theorem 8.2.1. *Under Assumptions B_0' and C_0, and if g is a globally Lipschitz function, there exists a unique mild solution $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ to equation (8.9). Moreover, the map $(t,x) \mapsto u(t,x)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^d$ to $L^2(\Omega)$ and for all $T > 0$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(u(t,x)^2) < \infty.$$

Proof. Following the scheme of the proof of Theorem 13 in [15], let us define recursively the sequence

$$u^{(0)}(t,x) = 0, \qquad u^{(n+1)}(t,x) = \int_0^t ds \int_{\mathbb{R}^d} G(s,dz)\, g(u^{(n)}(t-s,x-z)) + \int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dz_1)\, G(t-s,x_1-z_1,x_2).$$

From this, we have the following recurrence relation for $n \geq 1$:

$$u^{(n+1)}(t,x) - u^{(n)}(t,x) = \int_0^t ds \int_{\mathbb{R}^d} G(s,dz)\, \left(g(u^{(n)}(t-s,x-z)) - g(u^{(n-1)}(t-s,x-z))\right),$$

since the stochastic term in the above definition does not depend on n. We can therefore apply the argument of [15] to conclude that the sequence $(u^{(n)})$ converges to the solution of equation (8.10). The only difference comes in the evaluation of the first term of the sequence:

$$u^{(1)}(t,x) = \int_0^t ds \int_{\mathbb{R}^d} G(s,dz)\, g(0) + \int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dz_1)\, G(t-s,x_1-z_1,x_2).$$

To show that the recurrence in [15] works, we need to show that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(u^{(1)}(t,x)^2) < \infty. \tag{8.11}$$

Let us compute

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}(u^{(1)}(t,x)^2) &\leq 2 \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(\int_0^t ds\, G(s,\mathbb{R}^d)\, g(0)\right)^2 \\ &+ 2 \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\left(\left(\int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dz_1)\, G(t-s,x_1-z_1,x_2)\right)^2\right). \end{aligned}$$

Since when $a = 0$,

$$\int_0^t ds\, G(s,\mathbb{R}^d) = \int_0^t ds\, \mathcal{F}G(s,0) = \int_0^t ds\, s = \frac{t^2}{2},$$

and when $a \neq 0$,

$$\int_0^t ds\, G(s,\mathbb{R}^d) = \int_0^t ds\, e^{-as} \frac{\sinh |a| s}{|a|} \leq \frac{t\,(1 \wedge e^{2|a|t})}{|a|},$$

we obtain that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(\int_0^t ds\, G(s,\mathbb{R}^d)\, g(0)\right)^2 < \infty.$$

On the other hand, using the isometry (6.13) and Lemma 7.3.1, we have

$$\begin{aligned} &\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\left(\left(\int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dz_1)\, G(t-s,x_1-z_1,x_2)\right)^2\right) \\ &= \sup_{(t,x_2)\in[0,T]\times\mathbb{R}} \int_0^t ds \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\, \mathcal{F}_1 G(t-s,\xi_1,x_2)^2 \\ &\leq C_7(T) \int_{\mathbb{R}^{d-1}} \mu(d\xi_1)\, \frac{1+\ln\left(\sqrt{1+|\xi_1|^2}\right)}{\sqrt{1+|\xi_1|^2}} < \infty, \end{aligned}$$

by Assumption B_0', so estimate (8.11) is satisfied and the theorem is proven. $\square$

Note finally that because of technical difficulties, the equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2a\,\dfrac{\partial u}{\partial t}(t,x) - \Delta u(t,x) \\[2ex] \qquad = g(u(t,x)) + h(u(t,x))\ \dot{F}(t,x_1)\ \delta_0(x_2), \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\[2ex] u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0, \qquad x \in \mathbb{R}^d, \end{cases}$$

has not been studied. Contrary to the case of a non-linear term restricted to the hyperplane $x_2 = 0$, the Picard's iteration scheme used in [15] could not be applied, mainly because the following term

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}((u^{(n+1)}(t,x) - u^{(n)}(t,x))^2)$$

could not be estimated with respect to the preceding term in the sequence, without the strong assumption that μ is finite (that is, Γ is regular). The problem comes essentially from the supremum, which has in particular to be taken over all $x_2 \in \mathbb{R}$.

Chapter 9

Linear heat equation in $\mathbb{R}^d$ driven by noise on a k-plane

Let us consider the following parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \frac{1}{2}\,\Delta u(t,x) = \dot{F}(t,x_1)\,\delta_0(x_2), & t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\ u(0,x) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{9.1}$$

where $\dot{F}$ is the spatially homogeneous noise on the k-plane $\mathbb{R}^k \times \{0\}$ considered in Chapter 6.

Let us recall that in the case of a spatially homogeneous noise in $\mathbb{R}^d$, the optimal condition on the spectral measure μ of the noise which guarantees the existence of a real-valued solution is the same for both parabolic and hyperbolic equations (see [15, 30]). We shall see in this chapter that the situation is completely different in the case of a noise concentrated on a k-plane, because it is the regularity of the Green kernel that plays a crucial role, which was not the case for a spatially homogeneous noise on $\mathbb{R}^d$.

Let us go first over some key points from Chapter 5.

9.1 Existence of a weak solution

The first step of our analysis consists, as before, in giving a weak formulation to equation (9.1); a weak solution of equation (9.1) is a process $u = \{u(t),\ t \in \mathbb{R}_+\}$ with values in $\mathcal{S}'(\mathbb{R}^d)$ such that $\mathbb{P} - a.s.$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto \langle u(t), \varphi \rangle$ is continuous on $\mathbb{R}_+$ and satisfies, for all $t \in \mathbb{R}_+$,

$$\begin{cases} \langle u(t), \varphi \rangle - \dfrac{1}{2} \displaystyle\int_0^t ds\ \langle u(s), \Delta\varphi \rangle = F_t(\varphi(\cdot, 0)), \\ \langle u(0), \varphi \rangle = 0. \end{cases} \tag{9.2}$$

Remark 9.1.1. As for the hyperbolic equation, this equation can also be interpreted when $k = d-1$ as the weak formulation of the following classical equation in the upper half space:

$$\frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\Delta u(t,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}_+,$$

with the stochastic boundary condition

$$\frac{\partial u}{\partial x_2}(t,x_1,0) = \dot{F}(t,x_1).$$

We then consider the Green kernel of this equation, which is the solution of

$$\frac{\partial G}{\partial t} - \frac{1}{2}\,\Delta G = 0, \qquad G(0) = \delta_0. \tag{9.3}$$

Its Fourier transform in x satisfies

$$\frac{\partial \mathcal{F}G}{\partial t}(t,\xi) + \frac{|\xi|^2}{2}\,\mathcal{F}G(t,\xi) = 0, \qquad \mathcal{F}G(0,\xi) = 1, \tag{9.4}$$

so

$$\mathcal{F}G(t,\xi) = e^{-\frac{|\xi|^2\,t}{2}}, \qquad t \in \mathbb{R}_+,\ \xi \in \mathbb{R}^d, \tag{9.5}$$

or equivalently,

$$G(t,x) = \frac{1}{(2\pi t)^{\frac{d}{2}}}\,e^{-\frac{|x|^2}{2t}}, \qquad t > 0,\ x \in \mathbb{R}^d, \tag{9.6}$$

and this implies that for all $t > 0$, $G(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)_+$ (that is, the space of non-negative functions in $\mathcal{S}(\mathbb{R}^d)$).

Certain properties of the Green kernel of the hyperbolic equation defined by equation (5.15) are also satisfied by the present Green kernel. In particular, Lemmas 5.4.2 and 5.4.3 are satisfied, as mentioned before. This explains why the condition on the spectral measure μ which guarantees the existence of a real-valued solution is the same for hyperbolic and parabolic equations in the case of a spatially homogeneous noise.

Following the development in Chapter 5, which led to Theorem 5.5.4, one can show the existence of a solution to equation (9.2), which is given by

$$\langle u(t),\varphi\rangle = \int_{[0,t]\times\mathbb{R}^k} M(ds,dx_1)\ (G(t-s)*\varphi)(x_1,0), \qquad t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{9.7}$$

where M is the worthy martingale measure defined in Section 6.1. The question of uniqueness is more delicate. For the hyperbolic equation, we used Lemma 5.5.3 for proving uniqueness, and more specifically the fact that for a given smooth initial *or* terminal condition, there exists a unique classical solution to equation (5.27). This time-reversal property of the solution of the hyperbolic equation does not hold in the case of the heat equation. Moreover, it is a well known fact that even for the classical heat equation, one needs to impose some restriction on the growth of the solution in order to obtain uniqueness (see for example [24]).

9.2 Existence of a real-valued solution

Before going into an analysis similar to that made in Chapter 7 for the hyperbolic equation, let us note that the analysis of Chapter 6 can be performed with a few modifications (since Lemma 5.4.1 is not satisfied, some estimates must be changed), but the results obtained are exactly the same as those obtained for the hyperbolic equation (6.5). We are going to see that, contrary to the case of the hyperbolic equation, these results are absolutely not optimal for the heat equation (9.2). The reason for this difference is that the solution of the heat equation does not belong in general to some Sobolev space $H^\beta(\mathbb{R}^{d-k})$ in the coordinate x_2, even when there exists a real-valued solution.

Let us then consider the Fourier transform of G in the first k coordinates of x, which can be easily deduced from (9.5):

$$\mathcal{F}_1 G(t,\xi_1,x_2) = e^{-\frac{|\xi_1|^2\, t}{2}}\ \frac{1}{(2\pi t)^{\frac{d-k}{2}}}\ e^{-\frac{|x_2|^2}{2t}}. \tag{9.8}$$

We will need the following estimate on $\mathcal{F}_1 G$.

Lemma 9.2.1. *For all $T>0$ and $\varepsilon>0$, there exist $C(T,\varepsilon)>0$ and a function P with polynomial growth such that*

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \le C(T,\varepsilon)\ P(\xi_1)\ e^{-2\,\varepsilon\,|\xi_1|},$$

for all $t\in[0,T]$, $\xi_1\in\mathbb{R}^k$ and $x_2\in\mathbb{R}^{d-k}$ such that $|x_2|\ge\varepsilon$.

Proof. If $|\xi_1|\ge 1$, then

$$\begin{aligned}\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 &= \int_0^t ds\ e^{-|\xi_1|^2\,s}\ \frac{1}{(2\pi s)^{d-k}}\ e^{-\frac{|x_2|^2}{s}}\\ &\le \int_0^\infty ds\ e^{-|\xi_1|^2\,s}\ \frac{1}{(2\pi s)^{d-k}}\ e^{-\frac{|x_2|^2}{s}}.\end{aligned}$$

Using now [5, formula I.5.34], we obtain that this last expression is equal to

$$\frac{1}{(2\pi)^{d-k}}\ 2\left(\frac{|\xi_1|}{|x_2|}\right)^{d-k-1} K_{d-k-1}(2\,|x_2|\,|\xi_1|),$$

where K_ν is the modified Bessel function of the second kind and of order ν (see Appendix B). By estimates (B.1) and (B.2), there exists $C>0$ such that for all $r>0$,

$$K_\nu(r) \le \begin{cases} C\ e^{-r}\ \ln\left(\frac{1}{r}\right) & \text{if } \nu = 0,\\[2ex] C\ \dfrac{e^{-r}}{r^{|\nu|}} & \text{if } \nu\neq 0,\end{cases}$$

so we obtain that for $|\xi_1|\ge 1$ and $|x_2|\ge\varepsilon$,

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \le \frac{1}{(2\pi)^{d-k}}\ 2\,C\ e^{-2\varepsilon\,|\xi_1|}\ \ln\left(\frac{1}{2\varepsilon}\right),$$

when $k = d-1$, and

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \le \frac{1}{(2\pi)^{d-k}}\ 2\ \left(\frac{|\xi_1|}{\varepsilon}\right)^{d-k-1}\ C\ \frac{e^{-2\varepsilon\,|\xi_1|}}{(2\varepsilon)^{d-k-1}},$$

when $k < d-1$. If $|\xi_1| \le 1$, then

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \le \int_0^t ds\ \frac{1}{(2\pi s)^{d-k}}\ e^{-\frac{|x_2|^2}{s}} \le \int_0^T ds\ \frac{1}{(2\pi s)^{d-k}}\ e^{-\frac{\varepsilon^2}{s}} < \infty,$$

and the proof ends like the proof of Lemma 5.4.1. □

We are going to prove now, following the scheme of Section 7.2, that without any additional assumption on the spectral measure μ of the noise, there exists a real-valued process X defined outside the k-plane $x_2 = 0$ which is the weak solution of equation (9.2). The only restriction here is that we assume that the covariance Γ is non-negative on $\mathbb{R}^k$, in order to use Theorem 6.2.1. Note the strong difference with the case of the hyperbolic equation, for which there never exists a real-valued solution when $k < d-1$, and even in the case $k = d-1$, the existence of such a real-valued solution is subject to an important restriction (namely Assumption B_0).

We first have the following two lemmas.

Lemma 9.2.2. *For $(t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}$ fixed, the function $\phi_{t,x_1,x_2} : [0,t] \to O'_C(\mathbb{R}^k)_+$ defined by*

$$\phi_{t,x_1,x_2}(s,\cdot) = G(t-s, x_1 - \cdot, x_2), \qquad s \in [0,t],$$

belongs to H_t.

Proof. We use here Theorem 6.2.1; since $\phi_{t,x_1,x_2}(s,\cdot) \in O'_C(\mathbb{R}^k)_+$ for a fixed $s \in [0,t]$ and

$$\mathcal{F}_1\phi_{t,x_1,x_2}(s,\xi_1) = \mathcal{F}_1 G(t-s,-\xi_1,x_2)\ \chi_{x_1}(\xi_1)$$

is a Borel-measurable function, it suffices then to check that $\|\phi_{t,x_1,x_2}\|_t < \infty$:

$$\begin{aligned}\|\phi_{t,x_1,x_2}\|_t^2 &= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ \mathcal{F}_1 G(t-s,-\xi_1,x_2)^2 \\ &\le C(t,|x_2|) \int_{\mathbb{R}^k} \mu(d\xi_1)\ P(\xi_1)\ e^{-2\,|x_2|\,|\xi_1|},\end{aligned}$$

by Lemma 9.2.1. Since μ is a tempered measure and P has polynomial growth, the above expression is finite for all $x_2 \neq 0$, so the lemma is proven. □

Lemma 9.2.3. *Let M be the worthy martingale measure defined in Section 6.1. The real-valued process $X = \{X(t,x_1,x_2),\ (t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}\}$ defined by*

$$X(t,x_1,x_2) = \int_{[0,t]\times\mathbb{R}^k} M(ds,dz_1)\ G(t-s,x_1-z_1,x_2), \qquad (t,x_1,x_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\},$$

is a centered Gaussian process whose covariance is given by

$$\begin{aligned}&\mathbb{E}(X(t,x_1,x_2)\ X(s,y_1,y_2))\\ &= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^{t\wedge s} dr\ \mathcal{F}_1G(t-r,-\xi_1,x_2)\ \mathcal{F}_1G(s-r,-\xi_1,y_2)\ \chi_{x_1-y_1}(\xi_1),\end{aligned} \tag{9.9}$$

and such that the map $(t,x_1,x_2) \mapsto X(t,x_1,x_2)$ *is continuous from* $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}$ *to* $L^2(\Omega)$*.*

Remark 9.2.4. This result and [42, prop. 3.6 and cor. 3.8] imply that the process X admits a modification $\tilde{X}$ such that the map $(t,x_1,x_2,\omega) \mapsto \tilde{X}(t,x_1,x_2,\omega)$ is jointly measurable. We will implicitely consider this modification in the following.

Proof of Lemma 9.2.3 By Lemma 9.2.2, the process X is well defined. As in the case of the hyperbolic equation, the fact that X is a centered Gaussian process with the covariance given above follows easily from the isometry (6.13) and since μ and $\mathcal{F}_1G$ are symmetric in ξ_1, (9.9) is equal to

$$\int_{\mathbb{R}^{d-1}} \mu(d\xi_1) \int_0^{t\wedge s} dr\ \mathcal{F}_1G(t-r,-\xi_1,x_2)\ \mathcal{F}_1G(s-r,-\xi_1,y_2)\ \cos(\xi_1\cdot(x_1-y_1)),$$

so X is real-valued.

In order to show that the map $(t,x_1,x_2) \mapsto X(t,x_1,x_2)$ is continuous from $\mathbb{R}_+\times\mathbb{R}^k\times\mathbb{R}^{d-k}\backslash\{0\}$ to $L^2(\Omega)$, we show that for all $T>0$ and $R>\varepsilon>0$, it is continuous from $[0,T]\times\mathbb{R}^k\times K(R,\varepsilon)$ to $L^2(\Omega)$, where

$$K(R,\varepsilon) = \{x_2 \in \mathbb{R}^{d-k} \text{ such that } R \geq |x_2| \geq \varepsilon\}.$$

And we do this in two steps, showing first that there exists $C(T,R,\varepsilon)>0$ such that

$$\mathbb{E}((X(t,y_1,y_2)-X(t,x_1,x_2))^2) \leq C(T,R,\varepsilon)\ (|y_1-x_1|^2+|y_2-x_2|^2), \tag{9.10}$$

for all $t\in[0,T]$, $x_1,\ y_1\in\mathbb{R}^k$ and $x_2,\ y_2\in K(R,\varepsilon)$, which implies that the map $(x_1,x_2)\mapsto X(t,x_1,x_2)$ is L^2-continuous in $\mathbb{R}^k\times K(R,\varepsilon)$, uniformly in $t\in[0,T]$. The second step consists simply in showing that for fixed $(x_1,x_2)\in\mathbb{R}^k\times K(R,\varepsilon)$, the map $t\mapsto X(t,x_1,x_2)$ is L^2-continuous.

We begin by establishing (9.10). We have

$$\begin{aligned}&\mathbb{E}((X(t,y_1,y_2)-X(t,x_1,x_2))^2)\\ &\leq\ 2\,(\mathbb{E}((X(t,y_1,y_2)-X(t,x_1,y_2))^2)+\mathbb{E}((X(t,x_1,y_2)-X(t,x_1,x_2))^2),\end{aligned} \tag{9.11}$$

so we can handle the two terms separately. Let us then compute

$$\begin{aligned}&\mathbb{E}((X(t,y_1,y_2)-X(t,x_1,y_2))^2)\\ &= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ \mathcal{F}_1G(t-s,-\xi_1,y_2)^2\ |\chi_{y_1}(\xi_1)-\chi_{x_1}(\xi_1)|^2.\end{aligned}$$

Since

$$|\chi_{y_1}(\xi_1) - \chi_{x_1}(\xi_1)|^2 = \left| \int_{x_1\cdot\xi_1}^{y_1\cdot\xi_1} dr\ e^{ir} \right|^2 \leq |\xi_1|^2\ |y_1 - x_1|^2,$$

we obtain by Lemma 9.2.1 that

$$\mathbb{E}((X(t,y_1,y_2) - X(t,x_1,y_2))^2) \leq C(T,\varepsilon) \int_{\mathbb{R}^k} \mu(d\xi_1)\ P(\xi_1)\ |\xi_1|^2\ e^{-2\,\varepsilon\,|\xi_1|}\ |y_1 - x_1|^2,$$

which gives the desired result for the first term of (9.11), since

$$\int_{\mathbb{R}^k} \mu(d\xi_1)\ P(\xi_1)\ |\xi_1|^2\ e^{-2\,\varepsilon\,|\xi_1|} < \infty.$$

Let us now consider the second term:

$$\begin{aligned}
&\mathbb{E}((X(t,x_1,y_2) - X(t,x_1,x_2))^2) \\
&= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ (\mathcal{F}_1 G(t-s,-\xi_1,y_2) - \mathcal{F}_1 G(t-s,-\xi_1,x_2))^2 \\
&= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ e^{-|\xi_1|^2\ s}\ \frac{1}{(2\pi s)^{d-k}} \left(e^{-\frac{|x_2|^2}{2s}} - e^{-\frac{|y_2|^2}{2s}} \right)^2.
\end{aligned}$$

Suppose without loss of generality that $|y_2| \geq |x_2|$. Then

$$\left(e^{-\frac{|y_2|^2}{2s}} - e^{-\frac{|y_2|^2}{2s}} \right)^2 = e^{-\frac{|x_2|^2}{s}} \left(e^{-\frac{|y_2|^2 - |x_2|^2}{2s}} - 1 \right)^2.$$

Since

$$0 \leq 1 - e^{-x} \leq x, \qquad \forall x \geq 0,$$

and

$$\frac{|y_2|^2 - |x_2|^2}{2s} = \frac{(|y_2| + |x_2|)\ (|y_2| - |x_2|)}{2s} \leq \frac{R}{s}\ |y_2 - x_2|,$$

we obtain that

$$\begin{aligned}
&\mathbb{E}((X(t,x_1,y_2) - X(t,x_1,x_2))^2) \\
&= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ e^{-|\xi_1|^2\ s}\ \frac{1}{(2\pi s)^{d-k}}\ e^{-\frac{|x_2|^2}{s}}\ \frac{R^2}{s^2}\ |y_2 - x_2|^2.
\end{aligned}$$

By estimates similar to those carried out in the proof of Lemma 9.2.1, we obtain that there exist $\tilde{C}(T,\varepsilon) > 0$ and a function $\tilde{P}$ with polynomial growth such that

$$\int_0^t ds\ e^{-|\xi_1|^2\ s}\ \frac{1}{(2\pi s)^{d-k}}\ e^{-\frac{|x_2|^2}{s}}\ \frac{1}{s^2} \leq \tilde{C}(T,\varepsilon)\ \tilde{P}(\xi_1)\ e^{-2\,\varepsilon\,|\xi_1|}, \qquad \forall t \in [0,T],\ \xi_1 \in \mathbb{R}^k,\ |x_2| \geq \varepsilon,$$

so we have the bound that we wanted for the second term of (9.11), because

$$\int_{\mathbb{R}^k} \mu(d\xi_1)\ \tilde{P}(\xi_1)\ e^{-2\,\varepsilon\,|\xi_1|} < \infty.$$

Finally, we need to prove the L^2-continuity in t; consider $(t, x_1, x_2) \in [0,T] \times \mathbb{R}^k \times K(R,\varepsilon)$ fixed and $h \geq 0$:

$$\begin{aligned}
&\mathbb{E}((X(t+h, x_1, x_2) - X(t, x_1, x_2))^2) \\
&\quad = \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds\ (\mathcal{F}_1 G(s+h, -\xi_1, x_2) - \mathcal{F}_1 G(s, -\xi_1, x_2))^2 \qquad (9.12)\\
&\quad\quad + \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^h ds\ \mathcal{F}_1 G(s, -\xi_1, x_2)^2. \qquad (9.13)
\end{aligned}$$

Since for $\xi_1 \in \mathbb{R}^k$ and $|x_2| \geq \varepsilon$ fixed,

$$\mathcal{F}_1 G(s+h, -\xi_1, x_2) - \mathcal{F}_1 G(s, -\xi_1, x_2) \underset{h \to 0}{\to} 0$$

for all $s \in [0,t]$ and since for all $h \leq h_0$,

$$\begin{aligned}
&\sup_{s \in [0,t]} (\mathcal{F}_1 G(s+h, -\xi_1, x_2) - \mathcal{F}_1 G(s, -\xi_1, x_2))^2 \\
&\leq 2 \sup_{s \in [0,t]} \left(e^{-|\xi_1|^2 (s+h)} \frac{1}{(2\pi(s+h))^{d-k}} e^{-\frac{|x_2|^2}{s+h}} + e^{-|\xi_1|^2 s} \frac{1}{(2\pi s)^{d-k}} e^{-\frac{|x_2|^2}{s}} \right) \\
&\leq 4 \sup_{s \in [0, t+h_0]} \frac{1}{(2\pi s)^{d-k}} e^{-\frac{\varepsilon^2}{s}} < \infty,
\end{aligned}$$

we obtain by the dominated convergence theorem that

$$\int_0^t ds\ (\mathcal{F}_1 G(s+h, -\xi_1, x_2) - \mathcal{F}_1 G(s, -\xi_1, x_2))^2 \underset{h \to 0}{\to} 0.$$

Moreover, for all $h \leq h_0$,

$$\begin{aligned}
&\int_0^t ds\ (\mathcal{F}_1 G(s+h, -\xi_1, x_2) - \mathcal{F}_1 G(s, -\xi_1, x_2))^2 \\
&\leq 4 \int_0^{t+h} ds\ \mathcal{F}_1 G(s, -\xi_1, x_2)^2 \\
&\leq 4\, C(t+h_0, \varepsilon)\ P(\xi_1)\ e^{-2\,\varepsilon\,|\xi_1|},
\end{aligned}$$

by Lemma 9.2.1, so using once again the dominated convergence theorem together with the fact that

$$\int_{\mathbb{R}^k} \mu(d\xi_1)\ P(\xi_1)\ e^{-2\,\varepsilon\,|\xi_1|} < \infty,$$

we obtain that (9.12) converges to 0 as $h \to 0$. Consider now (9.13):

$$\begin{aligned}
\int_0^h ds\ \mathcal{F}_1 G(s, -\xi_1, x_2)^2 &= \int_0^h ds\ e^{-|\xi_1|^2\, s} \frac{1}{(2\pi s)^{d-k}} e^{-\frac{|x_2|^2}{s}} \\
&\leq \int_0^h ds\ e^{-|\xi_1|^2\, s} \frac{1}{(2\pi s)^{d-k}} e^{-\frac{|x_2|^2}{2s}} e^{-\frac{|x_2|^2}{2h}} \\
&\leq C(h_0, \frac{\varepsilon}{\sqrt{2}})\ P(\xi_1)\ e^{-\sqrt{2}\,\varepsilon\,|\xi_1|}\ e^{-\frac{\varepsilon^2}{2h}}
\end{aligned}$$

for all $h \leq h_0$ by Lemma 9.2.1. So (9.13) is less than or equal to

$$C(h_0, \frac{\varepsilon}{\sqrt{2}}) \int_{\mathbb{R}^k} \mu(d\xi_1)\ P(\xi_1)\ e^{-\sqrt{2}\,\varepsilon\,|\xi_1|}\ e^{-\frac{\varepsilon^2}{2h}}$$

which converges to 0 as $h \to 0$, since

$$\int_{\mathbb{R}^k} \mu(d\xi_1)\ P(\xi_1)\ e^{-\sqrt{2}\,\varepsilon\,|\xi_1|} < \infty.$$

This shows the right-continuity in t of the process X (in $L^2(\Omega)$). The left-continuity follows from the same argument as in the proof of Lemma 6.3.3, and this completes the proof. □

We can now state the following existence theorem.

Theorem 9.2.5. *Let u be the solution of equation (9.2). There exists then a real-valued centered Gaussian process $X = \{X(t, x_1, x_2), (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k} \backslash \{0\}\}$ whose covariance is given by formula (9.9), such that the map $(t, x_1, x_2) \mapsto X(t, x_1, x_2)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k} \backslash \{0\}$ to $L^2(\Omega)$ and*

$$\langle u(t), \varphi \rangle = \int_{\mathbb{R}^d} dx\ X(t, x)\ \varphi(x), \qquad \mathbb{P} - a.s.,$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that supp $\varphi \subset \mathbb{R}^k \times \mathbb{R}^{d-k} \backslash \{0\}$.

Proof. The first part of the theorem follows directly from Lemmas 9.2.2 and 9.2.3. The proof of the last equality follows then exactly the argument of the proof of Lemma 7.2.4. □

Remark 9.2.6. This theorem implies in particular that there exists a function-valued solution even when the noise is white, which was the case studied in [61] (for $k = d-1$). It is therefore clear that the results obtained in Chapters 2 and 3 are not optimal for the existence of a real-valued weak solution of the heat equation driven by boundary noise in a bounded domain: concerning this subject, see the extended analysis in [61] of parabolic partial differential equations driven by white boundary noises.

One question now remains: under some additional assumption on μ, does there exist a real-valued solution of equation (9.2) which is defined for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$? The next theorem shows that the answer is negative.

Theorem 9.2.7. *Let u be the solution of equation (9.2). There does not exist then a real-valued square integrable process $X = \{X(t, x),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ such that the map $(t, x) \mapsto X(t, x)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}^d$ to $L^2(\Omega)$ and*

$$\langle u(t), \varphi \rangle = \int_{\mathbb{R}^d} dx\ X(t, x)\ \varphi(x), \qquad \mathbb{P} - a.s., \forall t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Proof. Suppose that there exists a process X satisfying the above conditions. As in the proof of Theorem 7.2.5, let us define, for $t \in \mathbb{R}_+$ and $x_1 \in \mathbb{R}^k$, $\varphi^{(n)}_{x_1,0} = \delta_{(x_1,0)} * \psi_n \in \mathcal{S}(\mathbb{R}^d)$, where

(ψ_n) is a sequence of non-negative and compactly supported approximations of δ_0 in $\mathbb{R}^d$. By the assumptions made on X, we obtain that

$$\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,0}\rangle|^2) = \mathbb{E}(X(t,x_1,0)^2) < \infty. \tag{9.14}$$

On the other hand,

$$\begin{aligned} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,0}\rangle|^2) &= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds \, \left|\mathcal{F}_1(G(t-s) * \varphi^{(n)}_{x_1,0})(\xi_1, 0)\right|^2 \\ &= \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds \, \left|\mathcal{F}_1(G(s) * \varphi^{(n)}_{x_1,0})(\xi_1, 0)\right|^2. \end{aligned}$$

Moreover, by arguments similar to those used in the proof of Theorem 7.2.5,

$$\mathcal{F}_1(G(s) * \varphi^{(n)}_{x_1,0})(\xi_1, 0) \underset{n\to\infty}{\to} \mathcal{F}_1 G(s, \xi_1, 0) \; \chi_{x_1}(\xi_1),$$

for all $(s, \xi_1) \in \,]0, t] \times \mathbb{R}^k$. So Fatou's lemma tells us that

$$\lim_{n\to\infty} \mathbb{E}(|\langle u(t), \varphi^{(n)}_{x_1,0}\rangle|^2) \geq \int_{\mathbb{R}^k} \mu(d\xi_1) \int_0^t ds \, \mathcal{F}_1 G(s, \xi_1, 0)^2.$$

But since we have, for a fixed $\xi_1 \in \mathbb{R}^k$,

$$\int_0^t ds \, \mathcal{F}_1 G(s, \xi_1, 0)^2 = \int_0^t ds \, e^{-|\xi_1|^2 s} \frac{1}{(2\pi s)^{d-k}} = \infty,$$

the above expression is also infinite, which contradicts (9.14), so the theorem is proven. □

This theorem implies, among other things, that we will not be able to consider non-linear equations of the form

$$\frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\,\Delta u(t,x) = f(u(t,x_1,0))\;\delta_0(x_2) + g(u(t,x_1,0))\;\dot{F}(t,x_1)\;\delta_0(x_2).$$

On the other hand, it is possible to analyze equations of the form

$$\frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\,\Delta u(t,x) = h(u(t,x)) + \dot{F}(t,x_1)\;\delta_0(x_2),$$

because the solution u does not need to be defined on the k-plane in this case. Since this study has been performed quite extensively in [61] in the case of a white noise concentrated on a hyperplane (and even more generally on a C^∞ boundary), we will not go deeper into this analysis (remember that the boundary noise in [61] was interpreted as a stochastic boundary condition).

One could also notice that we have skipped the question of the Hölder regularity of the solution in the preceding section. Actually, looking at the proof of Lemma 9.2.3, one can already notice that (9.11) implies that the process X is $\mathbb{P}-a.s.$ locally Hölder-continuous in x with exponent $\gamma < 1$ on $\mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}$. But the regularizing property of the Green kernel implies much more, that is, the process X is $\mathbb{P}-a.s.$ C^∞ on $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{d-k}\backslash\{0\}$; see [61] for more details.

Finally, note that in the case of a *white* noise concentrated on a k-plane (that is, when $\mu(d\xi_1) = d\xi_1$), we can estimate the behavior in x_2 of $\mathbb{E}(X(t,x_1,x_2)^2)$ near the k-plane $x_2 = 0$:

$$\begin{aligned}
&\mathbb{E}(X(t,x_1,x_2)^2) \\
&= \int_0^t ds \int_{\mathbb{R}^k} d\xi_1 \, \mathcal{F}_1 G(s,\xi_1,x_2)^2 = \int_0^t ds \, \frac{1}{(2\pi s)^{d-k}} \, e^{-\frac{|x_2|^2}{s}} \int_{\mathbb{R}^k} d\xi_1 \, e^{-|\xi_1|^2 s} \\
&= \int_0^t ds \, \frac{1}{(2\pi s)^{d-k}} \, e^{-\frac{|x_2|^2}{s}} \left(\frac{4\pi}{s}\right)^{\frac{k}{2}} = C \int_0^t ds \, \frac{1}{s^{d-\frac{k}{2}}} \, e^{-\frac{|x_2|^2}{s}},
\end{aligned}$$

where $C = 2^{2k-d} \, \pi^{\frac{3k}{2}-d}$. Making now the change of variable $u^2 = \frac{|x_2|^2}{s}$, we obtain

$$\mathbb{E}(X(t,x_1,x_2)^2) = \frac{2\,C}{|x_2|^{2d-2-k}} \int_{\frac{|x_2|}{\sqrt{t}}}^{\infty} du \, u^{2d-3-k} \, e^{-u^2} \underset{|x_2|\to 0}{\sim} \frac{1}{|x_2|^{2d-2-k}}.$$

This generalizes the estimate obtained in the case of a noise on an hyperplane in [61] to the case of a noise on a lower-dimensional plane, and shows that the solution has not an L^2-behavior near the k-plane $x_2 = 0$.

Chapter 10

Perspectives

Let us first make a general comment and observe the relationships between some results obtained in this dissertation. In particular, let us look simultaneously at conditions (3.11) and B_0. Once we realize that the sequence (a_l) of Chapter 3 is nothing but the "spectral measure" of the noise in a discrete case (because the a_l are the Fourier coefficients of the covariance Γ), the similarity between these two conditions becomes evident. Apart from this, Theorems 3.3.3 and 6.4.3 state that these are both necessary (and nearly sufficient) conditions for the existence of an L^2-type solution of the heat or the wave equation driven by a boundary noise term with such a spectral measure. Moreover, their reformulation into a condition on the covariance gives the same result when $d = 2$: see Sections 3.4, 7.5 and below. On the other hand, one can also appreciate the similarity between conditions (3.10) and B_0'. This gives us the intuition that there should be some generalization of the results obtained in this dissertation for equations driven by noises concentrated on manifolds of various shapes.

A first possible generalization of the results obtained in Chapter 3 and Appendix C should be the following: for the heat or the wave equation in a bounded domain D driven by noise concentrated on a manifold S which is part of the boundary of the domain, there exists a unique weak solution to the equation (in the sense given in (2.4)) with values in $L^2(D)$ if and only if the covariance Γ_S of the noise can be represented by a trace-class linear operator Q_S on the Sobolev space $H^{\frac{1}{2}}(S)$. The question is: does there exist a general argument for this (perhaps using the general theory developed by G. Da Prato and J. Zabczyk in [18]) and what is the intuition behind it? One could also ask if the situation remains the same when the manifold S is part of the interior of the domain (beginning by studying, as in Section 3.5, the equation driven by noise concentrated on a sphere of radius r_0 less than 1, interior to the ball $B(0,1)$).

Let us now give a possible generalization of the results obtained in Chapter 7. For this, we first make more explicit the connections between the results obtained for the wave equation driven by noise on a sphere in Chapter 3 and that driven by noise on a hyperplane in Chapter 7. In Chapter 3, we have seen that if there exists an L^2-valued solution to the hyperbolic equation in the unit disc driven by noise concentrated on the unit circle S^1 with isotropic covariance f, then the following condition is satisfied:

$$\int_0^\pi d\theta\ f(\theta)\ \ln\left(\frac{1}{\theta}\right) < \infty,$$

and this condition was shown to be nearly sufficient. A first improvement of this result would be to consider when there exists a *real*-valued solution to this equation and see if the same condition appears, which seems plausible. Note however that as already mentioned in Remark 3.2.4, the solution of the equation probably explodes at the center of the sphere, because the entire influence of the noise on the sphere reaches this point at the same time. If we want therefore the solution to be real-valued in the general setting of a noise concentrated on a manifold, we should not expect this to be true for every point in space (and we will also probably have to distinguish the two cases where the solution is defined only outside the manifold or also on the manifold itself, as in the case of a noise on a hyperplane).

On the other hand, in Chapter 7, we have seen that if there exists a real-valued process defined outside the line $x_2 = 0$ which is the weak solution of the wave equation in $\mathbb{R}^2$ driven by noise concentrated on this line with spatially homogenous covariance f on the line, then the following condition is satisfied:

$$\int_0^1 dr\ f(r)\ \ln\left(\frac{1}{r}\right) < \infty,$$

whose similarity with the above condition has not to be proven. For the equation in $\mathbb{R}^d$ (with $d \geq 3$) driven by noise on a hyperplane, the above condition simply becomes

$$\int_0^1 dr\ f(r) < \infty.$$

Although it has not been established that the same kind of condition appears in the case of a noise on a sphere, this is also quite plausible.

Moreover, the analysis of Chapter 7 shows that there never exists a real-valued solution when the noise is concentrated on a k-plane of dimension $k = d-2$, and seemingly neither when $k < d-2$.

One clear improvement of this work would then be to generalize these results to hyperbolic equations in $\mathbb{R}^d$ driven by noise concentrated on a general manifold. The first problem is the following: our results are expressed for noises with some rotational or spatial homogeneity. On a general manifold, such a homogeneity does not exist. Nevertheless, we can restrict ourselves to noises concentrated on a manifold S with covariance Γ_S given by

$$\Gamma_S(\varphi,\psi) = \int_S d\sigma(x) \int_S d\sigma(y)\ \varphi(x)\ f(\partial(x,y))\ \overline{\psi(y)}, \qquad \varphi,\psi \in \mathcal{S}(\mathbb{R}^d),$$

where σ is the uniform measure on S (induced by the Lebesgue measure on $\mathbb{R}^d$), $\partial(x,y)$ is the *geodesic distance* between two points x and y on the manifold S and f is a continuous function on $]0,\infty[$. We conjecture then that there exists a real-valued process defined outside the manifold S and up to a given time defined as the mininum of the radii of curvature of the manifold S, which is the solution of the hyperbolic equation in $\mathbb{R}^d$ driven by noise concentrated on S with covariance of the form given above, only if S is of dimension $d-1$ and the following condition is satisfied:

$$\begin{cases} \displaystyle\int_0^1 dr\ f(r)\ \ln\left(\frac{1}{r}\right) < \infty, & \text{when } d = 2, \\[2ex] \displaystyle\int_0^1 dr\ f(r) < \infty, & \text{when } d \geq 3. \end{cases}$$

Moreover, this condition should be shown to be nearly sufficient, in a sense to be made precise. We believe that this is achievable since the conditions obtained are all local, and therefore should not depend on the particular shape of the manifold considered.

Appendix A

Green kernel of the hyperbolic equation in $\mathbb{R}^d$

A.1 Expressions for the Green kernel

From (5.18) and by calculations similar to (7.1) and (7.31), we obtain the following explicit expressions for G when $d \leq 3$ and a, b are any real numbers. If $a^2 \geq b$, then:

when $d = 1$:

$$G(t,x) = \frac{e^{-at}}{2}\, I_0\left(\sqrt{(a^2-b)\,(t^2-x^2)}\right)\, 1_{\{|x| < t\}}, \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}, \tag{A.1}$$

when $d = 2$:

$$G(t,x) = \frac{e^{-at}}{2\pi}\, \frac{\cosh\left(\sqrt{(a^2-b)\,(t^2-|x|^2)}\right)}{\sqrt{t^2-|x|^2}}\, 1_{\{|x| < t\}}, \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^2, \tag{A.2}$$

when $d = 3$:

$$\begin{aligned} G(t,dx) \quad = \quad & \frac{e^{-at}}{4\pi}\left(\frac{1}{t}\,\delta_{\{|x| = t\}}(dx)\right. \\ & \left. +\sqrt{a^2-b}\, \frac{I_1\left(\sqrt{(a^2-b)\,(t^2-|x|^2)}\right)}{\sqrt{t^2-|x|^2}}\, 1_{\{|x| < t\}}\, dx\right), \qquad t \in \mathbb{R}_+, \end{aligned} \tag{A.3}$$

and if $a^2 < b$, then:

when $d = 1$:

$$G(t,x) = \frac{e^{-at}}{2}\, J_0\left(\sqrt{(b-a^2)\,(t^2-x^2)}\right)\, 1_{\{|x| < t\}}, \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}, \tag{A.4}$$

when $d = 2$:

$$G(t,x) = \frac{e^{-at}}{2\pi}\, \frac{\cos\left(\sqrt{(b-a^2)\,(t^2-|x|^2)}\right)}{\sqrt{t^2-|x|^2}}\, 1_{\{|x| < t\}}, \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^2, \tag{A.5}$$

when $d = 3$:

$$G(t, dx) = \frac{e^{-at}}{4\pi} \left(\frac{1}{t} \, \delta_{\{|x| = t\}}(dx) - \sqrt{b - a^2} \, \frac{J_1\left(\sqrt{(b-a^2)\,(t^2 - |x|^2)}\right)}{\sqrt{t^2 - |x|^2}} \, 1_{\{|x| < t\}} \, dx \right), \qquad t \in \mathbb{R}_+, \quad \text{(A.6)}$$

where J_ν and I_ν are the regular and modified Bessel functions of the first kind and of order ν (see Appendix B). For the calculation in the case $d = 3$, see [45, formulas I.14.46 and I.18.33] for the regular part of G; the singular part can be computed separately in the case $a = b = 0$.

Note moreover that in the case where $a = 0$ (namely the case of the Klein-Gordon equation), (A.4) and (A.6) are formulas (7.3.88) and (11.1.19) in [25], respectively.

On the other hand, when d is any positive natural number and $a = b = 0$, we also have the following expressions for G (see [37, p.281]):

when $d = 1$:

$$G(t, x) = \frac{1}{2} \, 1_{\{|x| < t\}}, \qquad t \in \mathbb{R}_+, \; x \in \mathbb{R}, \quad \text{(A.7)}$$

when d is even:

$$G(t, \varphi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{d-2}{2}} \left(t^{d-1} \int_{|x|<1} dx \, \frac{\varphi(tx)}{\sqrt{1 - |x|^2}} \right), \qquad t \in \mathbb{R}_+, \; \varphi \in \mathcal{S}(\mathbb{R}^d), \quad \text{(A.8)}$$

when d is odd and $d \geq 3$:

$$G(t, \varphi) = \frac{1}{2} \, \frac{1}{(2\pi)^{\frac{d-1}{2}}} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{d-3}{2}} \left(t^{d-2} \int_{|x|=1} d\sigma(x) \; \varphi(tx) \right), \qquad t \in \mathbb{R}_+, \; \varphi \in \mathcal{S}(\mathbb{R}^d). \quad \text{(A.9)}$$

Let us write these expressions more explicitely when $d = 4$:

$$G(t, \varphi) = \frac{1}{4\pi^2 t^2} \int_{|x|<t} dx \, \frac{3\varphi(x) + \nabla\varphi(x) \cdot x}{\sqrt{t^2 - |x|^2}}, \qquad t \in \mathbb{R}_+, \; \varphi \in \mathcal{S}(\mathbb{R}^4), \quad \text{(A.10)}$$

and when $d = 5$:

$$G(t, \varphi) = \frac{1}{8\pi^2 t^2} \int_{|x|=t} d\sigma(x) \; (3\varphi(x) + \nabla\varphi(x) \cdot x), \qquad t \in \mathbb{R}_+, \; \varphi \in \mathcal{S}(\mathbb{R}^5). \quad \text{(A.11)}$$

From all these expressions, one can observe that $G(t, \cdot)$ is a *non-negative* distribution on $\mathbb{R}^d$ for all $t \in \mathbb{R}_+$ if and only if $d \leq 3$ and $a^2 \geq b$. Moreover, it is a measure if and only if $d \leq 3$.

Remark A.1.1. The reason for which we restricted our study in Chapter 5 to the two cases where either a and b are any real numbers and $d \leq 3$, or d is any positive natural number and $a = b = 0$, is that we need the above explicit expressions in order to verify property (5.16). Nevertheless, explicit expressions can certainly be computed in the general case and property (5.16) is likely to remain satisfied.

A.2 Green kernel restricted to a hyperplane

From (A.2), (A.3), (A.5) and (A.6), we deduce the following expressions for $G(t,\cdot,x_2)$ when $d \in \{2,3\}$ and a,b are any real numbers. Fix $t \in \mathbb{R}_+$ and $x_2 \in]-t,t[$. If $a^2 \geq b$, then:

when $d = 2$:

$$G(t,x_1,x_2) = \frac{e^{-at}}{2\pi} \frac{\cosh\left(\sqrt{(a^2-b)\ (t^2-x_2^2-x_1^2)}\right)}{\sqrt{t^2-x_2^2-x_1^2}}\ 1_{\left\{|x_1|<\sqrt{t^2-x_2^2}\right\}}, \tag{A.12}$$

when $d = 3$:

$$\begin{aligned} G(t,dx_1,x_2) \quad &= \quad \frac{e^{-at}}{4\pi}\left(\frac{1}{\sqrt{t^2-x_2^2}}\ \delta_{\left\{|x_1|=\sqrt{t^2-x_2^2}\right\}}(dx_1)\right. \\ &\left. +\sqrt{a^2-b}\ \frac{I_1\left(\sqrt{(a^2-b)\ (t^2-x_2^2-|x_1|^2)}\right)}{\sqrt{t^2-x_2^2-|x_1|^2}}\ 1_{\left\{|x_1|<\sqrt{t^2-x_2^2}\right\}}\ dx_1\right), \end{aligned} \tag{A.13}$$

and if $a^2 < b$, then:

when $d = 2$:

$$G(t,x_1,x_2) = \frac{e^{-at}}{2\pi} \frac{\cos\left(\sqrt{(b-a^2)\ (t^2-x_2^2-x_1^2)}\right)}{\sqrt{t^2-x_2^2-x_1^2}}\ 1_{\left\{|x_1|<\sqrt{t^2-x_2^2}\right\}}, \tag{A.14}$$

when $d = 3$:

$$\begin{aligned} G(t,dx_1,x_2) \quad &= \quad \frac{e^{-at}}{4\pi}\left(\frac{1}{\sqrt{t^2-x_2^2}}\ \delta_{\left\{|x_1|=\sqrt{t^2-x_2^2}\right\}}(dx_1)\right. \\ &\left. +\sqrt{b-a^2}\ \frac{J_1\left(\sqrt{(b-a^2)\ (t^2-x_2^2-|x_1|^2)}\right)}{\sqrt{t^2-x_2^2-|x_1|^2}}\ 1_{\left\{|x_1|<\sqrt{t^2-x_2^2}\right\}}\ dx_1\right). \end{aligned} \tag{A.15}$$

On the other hand, from (A.10) and (A.11), we deduce the following expressions for $G(t,\cdot,x_2)$ when $d = 4$ and $a = b = 0$:

$$G(t,\varphi_1,x_2) = H\left(\sqrt{t^2-x_2^2},\varphi_1\right)\ 1_{\{|x_2|\,<\,t\}}, \tag{A.16}$$

where

$$H(s,\varphi_1) = \frac{1}{4\pi^2 s^2}\int_{|x_1|<s} dx_1\ \frac{2\varphi_1(x_1)+\nabla_1\varphi_1(x_1)\cdot x_1}{\sqrt{s^2-|x_1|^2}}, \qquad s\in\mathbb{R}_+,\ \varphi_1\in\mathcal{S}(\mathbb{R}^3),$$

and when $d = 5$:

$$G(t,\varphi_1,x_2) = \left(H\left(\sqrt{t^2-x_2^2},\varphi_1\right) - \frac{1}{8\pi^2 t^2}\int_{|x_1|=\sqrt{t^2-x_2^2}} d\sigma(x_1)\ \varphi_1(x_1)\right)\ \frac{1}{t}\ 1_{\{|x_2|\,<\,t\}}, \tag{A.17}$$

where

$$H(s,\varphi_1) = \frac{1}{8\pi^2 s^2} \int_{|x_1|=s} d\sigma(x_1)\ \left(3\varphi_1(x_1) + \nabla_1\varphi_1(x_1)\cdot x_1\right), \qquad s \in \mathbb{R}_+,\ \varphi_1 \in \mathcal{S}(\mathbb{R}^4).$$

As before, we deduce from these expressions that $G(t,\cdot,x_2)$ is a non-negative measure on $\mathbb{R}^{d-1}$ for all $t \in \mathbb{R}_+$ and $x_2 \in \mathbb{R}$ if and only if $d \in \{2,3\}$ and $a^2 \geq b$. Moreover, it is a measure if and only if $d \in \{2,3\}$.

Remark A.2.1. The reason for which we restricted our study in Chapter 7 to the two cases where either $d \in \{2,3\}$ and a, b are any real numbers, or $d \in \{4,5\}$ and $a = b = 0$, is that we need the above explicit expressions in order to verify property (7.2). Nevertheless, explicit expressions can certainly be computed in the general case and property (7.2) is likely to remain satisfied.

Appendix B

Bessel functions

B.1 Definitions

Let $\nu \in \mathbb{R}$. We have the following definitions. The regular Bessel funtions of the first kind and of order ν are given by (see formula 9.1.10 in [1]):

$$J_\nu(r) = \left(\frac{r}{2}\right)^\nu \sum_{n\in\mathbb{N}} \frac{(-\frac{r^2}{4})^n}{n!\,\Gamma(\nu+n+1)}, \qquad r \in \mathbb{R}_+,$$

where Γ is the Gamma function defined in (3.2). The modified Bessel funtions of the first kind and of order ν are given by (see formula 9.6.10 in [1]):

$$I_\nu(r) = \left(\frac{r}{2}\right)^\nu \sum_{n\in\mathbb{N}} \frac{(\frac{r^2}{4})^n}{n!\,\Gamma(\nu+n+1)}, \qquad r \in \mathbb{R}_+.$$

Finally, for $\nu \in \mathbb{R}_+$, the modified Bessel funtions of the second kind and of order ν are given by (see formula 9.6.23 in [1]):

$$K_\nu(r) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})} \left(\frac{r}{2}\right)^\nu \int_1^\infty dt\; e^{-rt}\,(t^2-1)^{\nu-\frac{1}{2}}, \qquad r \in \mathbb{R}_+,$$

and $K_{-\nu}(r) = K_\nu(r)$ by formula 9.6.6 in [1].

Remark B.1.1. From these definitions, we directly see that I_ν and K_ν are *non-negative* functions, independently of the order ν considered. This fact is used repeatedly in this dissertation.

B.2 Estimates

We present here some estimates on J_0, I_0 and K_ν, with fixed $\nu \in \mathbb{R}$. First note that all these functions are continuous, except K_ν at the point $r = 0$.

Let us begin with K_ν: by formula 9.7.2 in [1], there exists $C > 0$ such that

$$\left| K_\nu(r) - \sqrt{\frac{\pi}{2r}} e^{-r} \right| \le \frac{C}{r^{\frac{3}{2}}}, \qquad \forall r \ge 1,$$

so there exists $C > 0$ such that

$$K_\nu(r) \leq C\, e^{-r}, \qquad \forall r \geq 1. \tag{B.1}$$

On the other hand, when $r \to 0$, we have by formulas 9.6.8 and 9.6.9 in [1]:

$$K_\nu(r) \sim \begin{cases} \ln\left(\frac{1}{r}\right) & \text{if } \nu = 0, \\ \dfrac{1}{r^{|\nu|}} & \text{if } \nu \neq 0, \end{cases} \tag{B.2}$$

Let us now consider I_0. By formula 9.6.16 in [1], I_0 admits the following integral representation for $r \in \mathbb{R}_+$:

$$I_0(r) = \frac{1}{\pi} \int_0^\pi \cosh(r\cos(t))\, dt,$$

so $I_0(0) = 1$ and I_0 is increasing on $\mathbb{R}_+$. By formula 9.7.1 in [1], there also exists $C > 0$ such that

$$\left| I_0(r) - \frac{e^r}{\sqrt{2\pi r}} \right| \leq \frac{C}{r^{3/2}}, \qquad \forall r > 0,$$

so there exists $C > 0$ such that

$$I_0(r) \leq C\, e^r, \qquad \forall r \geq 0. \tag{B.3}$$

Let us finally consider J_0. By formula 9.1.18 in [1], J_0 admits the following integral representation for $r \in \mathbb{R}_+$:

$$J_0(r) = \frac{1}{\pi} \int_0^\pi \cos(r\sin(t))\, dt,$$

so $|J_0(r)| \leq 1$ for all $r \in \mathbb{R}_+$, $J_0(0) = 1$ and J_0 is decreasing on $[0, 1]$. By formula 9.1.28 in [1], $J_0'(r) = -J_1(r)$, and by formula 9.2.1 in [1], there also exists $C > 0$ such that

$$\left| J_0(r) - \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi}{4}\right) \right| \leq \frac{C}{r^{3/2}}, \qquad \forall r > 0, \tag{B.4}$$

so there exists $C > 0$ such that

$$J_0(r)^2 \leq \frac{C}{\sqrt{1 + r^2}}, \qquad \forall r \geq 0, \tag{B.5}$$

which implies that

$$J_0(r)^2 \leq \frac{C}{r}, \qquad \forall r > 0. \tag{B.6}$$

Let us also mention the two following useful estimates.

Lemma B.2.1. *There exists $C > 0$ such that*

$$\frac{1}{R} \int_0^R dr\; r\; J_0(r)^2 \geq C, \qquad \forall R \geq 1.$$

Proof. Let us first prove the following:

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R dr\ r\ J_0(r)^2 = \frac{1}{\pi}. \tag{B.7}$$

To see this, note that

$$\begin{aligned}
&\left| r\ J_0(r)^2 - \frac{2}{\pi}\ \cos\left(\frac{\pi}{4} - r\right)^2 \right| \\
&= \left| \sqrt{r}\ J_0(r) - \sqrt{\frac{2}{\pi}}\ \cos\left(\frac{\pi}{4} - r\right) \right| \left| \sqrt{r}\ J_0(r) + \sqrt{\frac{2}{\pi}}\ \cos\left(\frac{\pi}{4} - r\right) \right| \\
&\leq \frac{C}{1+r},
\end{aligned} \tag{B.8}$$

by estimates (B.4) and (B.5). This implies that

$$\left| \int_0^R dr\ r\ J_0(r)^2 - \frac{2}{\pi} \int_0^R dr\ \cos\left(\frac{\pi}{4} - r\right)^2 \right| \leq C\ \ln(1+R),$$

But since

$$\int_0^R dr\ \cos\left(\frac{\pi}{4} - r\right)^2 = \frac{R}{2} + \frac{1 - \sin(\frac{\pi}{2} - 2R)}{4},$$

we obtain that (B.7) is true, which implies that for all $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\frac{1}{R} \int_0^R dr\ r\ J_0(r)^2 \geq \frac{1}{\pi} - \varepsilon, \qquad \forall R \geq R_0.$$

Suppose now that $R_0 > 1$ (otherwise, there is nothing left to prove). On $[1, R_0]$, the above function of R is greater than 0, so by the compactness of this interval, there exists $\delta > 0$ such that

$$\frac{1}{R} \int_0^R dr\ r\ J_0(r)^2 \geq \delta, \qquad \forall R \in [1, R_0].$$

The lemma is then satisfied with $C = (\frac{1}{\pi} - \varepsilon) \wedge \delta$. □

Lemma B.2.2. *There exists $C > 0$ such that for all $R \geq \pi$,*

$$\int_1^R dr\ J_0(r)^2 \geq \frac{\ln(R)}{\pi} - C.$$

Proof. For proving this, we use again (B.8) and obtain that there exists $C > 0$ such that for all $r \geq 1$,

$$\left| J_0(r)^2 - \frac{2}{\pi r}\ \cos\left(\frac{\pi}{4} - r\right)^2 \right| \leq \frac{C}{r^2}.$$

From this, we deduce that for $R \geq \pi$,

$$\left| \int_1^R dr\ J_0(r)^2 - \frac{2}{\pi} \int_1^R dr\ \frac{1}{r}\ \cos\left(\frac{\pi}{4} - r\right)^2 \right| \leq C\ (1 - \frac{1}{R}) \leq C, \tag{B.9}$$

Let us now compute

$$\int_1^R dr\ \frac{\cos\left(\frac{\pi}{4} - r\right)^2}{r},$$

observing first that

$$\int_1^R dr\ \frac{\cos\left(\frac{\pi}{4}-r\right)^2}{r} + \int_1^R dr\ \frac{\sin\left(\frac{\pi}{4}-r\right)^2}{r} = \ln(R),$$

but that on the other hand,

$$\left|\int_1^R dr\ \frac{\cos\left(\frac{\pi}{4}-r\right)^2}{r} - \int_1^R dr\ \frac{\sin\left(\frac{\pi}{4}-r\right)^2}{r}\right| = \left|\int_{1-\frac{\pi}{4}}^{R-\frac{\pi}{4}} dr\ \frac{\cos(r)^2}{r+\frac{\pi}{4}} - \int_{1+\frac{\pi}{4}}^{R+\frac{\pi}{4}} dr\ \frac{\cos(r)^2}{r-\frac{\pi}{4}}\right|.$$

Dividing these two integrals in three parts, we obtain that the above expression is less than or equal to

$$\begin{aligned}&\int_{1-\frac{\pi}{4}}^{1+\frac{\pi}{4}} dr\ \frac{1}{r+\frac{\pi}{4}} + \int_{1+\frac{\pi}{4}}^{R-\frac{\pi}{4}} dr\ \left(\frac{1}{r-\frac{\pi}{4}} - \frac{1}{r+\frac{\pi}{4}}\right) + \int_{R-\frac{\pi}{4}}^{R+\frac{\pi}{4}} dr\ \frac{1}{r-\frac{\pi}{4}} \\ = \quad & \ln(1+\frac{\pi}{2}) + \ln(R-\frac{\pi}{2}) - \ln(\frac{R}{1+\frac{\pi}{2}}) + \ln(\frac{R}{R-\frac{\pi}{2}}) = 2\ln(1+\frac{\pi}{2}).\end{aligned}$$

Therefore,

$$\int_1^R dr\ \frac{\cos\left(\frac{\pi}{4}-r\right)^2}{r} \geq \int_1^R dr\ \frac{\sin\left(\frac{\pi}{4}-r\right)^2}{r} - 2\ln(1+\frac{\pi}{2}),$$

and this implies that

$$\begin{aligned}\int_1^R dr\ \frac{\cos\left(\frac{\pi}{4}-r\right)^2}{r} \quad &\geq \quad \frac{1}{2}\left(\int_1^R dr\ \frac{\cos\left(\frac{\pi}{4}-r\right)^2}{r} + \int_1^R dr\ \frac{\sin\left(\frac{\pi}{4}-r\right)^2}{r} - 2\ln(1+\frac{\pi}{2})\right) \\ &= \quad \frac{\ln(R)}{2} - \ln(1+\frac{\pi}{2}).\end{aligned}$$

So finally, by (B.9),

$$\begin{aligned}\int_1^R dr\ J_0(r)^2 \quad &\geq \quad \frac{2}{\pi}\int_1^R dr\ \frac{1}{r}\ \cos\left(\frac{\pi}{4}-r\right)^2 - C \\ &\geq \quad \frac{\ln(R)}{\pi} - C - \ln(1+\frac{\pi}{2}),\end{aligned}$$

which proves the lemma. □

Appendix C

Linear equation driven by noise on one side of a hypercube

Let d be a natural number greater than 1, $D = [0, \pi]^d$ and $K = [0, \pi]^{d-1} \times \{0\}$ be the "bottom" side of the hypercube D (when $d = 3$, K is the square at the base of the cube D). In this chapter, we would like to study the existence of a weak solution to the hyperbolic equation (2.1) (in the sense defined in (2.4)), in the specific case where the domain D is the hypercube defined above and the noise considered is concentrated on K. For this, we follow an analysis similar to that of Chapter 3.

Let us first define the noise concentrated on K, considering a quite general covariance. Following an argument similar to that of Section 3.2, let us begin with a continuous and symmetric function $h : [-\pi, \pi]^{d-1} \to \mathbb{R}$ being non-negative definite on K, that is,

$$\sum_{i,j=1}^{m} c_i \, \overline{c_j} \, h(x^{(i)} - x^{(j)}) \geq 0, \qquad \forall m \geq 1, \ c_1, \ldots, c_m \in \mathbb{C}, \ x^{(1)}, \ldots, x^{(m)} \in K.$$

This function represents then the covariance of a Gaussian process indexed by the elements of K, which is moreover spatially homogeneous, that is, the covariance between two points x and y depends only on the vector $y - x$. Belonging to this class of functions are the following functions h:

$$h(x_1, \ldots, x_{d-1}) = \sum_{n_1, \ldots, n_{d-1} \in \mathbb{N}} a_{n_1, \ldots, n_{d-1}} \ \cos(n_1 x_1) \ \cdots \ \cos(n_{d-1} x_{d-1}), \quad (x_1, \ldots, x_{d-1}) \in K,$$

where

$$a_{n_1, \ldots, n_{d-1}} \geq 0 \quad \text{and} \quad \sum_{n_1, \ldots, n_{d-1} \in \mathbb{N}} a_{n_1, \ldots, n_{d-1}} < \infty.$$

Note that we cannot apply here the classical Bochner theorem to conclude that any covariance h has the preceding form, because K is not a group. Nevertheless, one can check that such h satisfies the required properties (using the formula $\cos(m(x - y)) = \cos(mx)\cos(my) + \sin(mx)\sin(my)$).

Following now the idea of Section 3.2, let us consider that the covariance of the noise concentrated on K is given by

$$\Gamma_K(\varphi,\psi) = \sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} a_{n_1,\dots,n_{d-1}}\, \Gamma_{n_1,\dots,n_{d-1}}(\varphi,\psi), \qquad \varphi,\psi \in C^\infty(K),$$

where $a_{n_1,\dots,n_{d-1}} \geq 0$ and

$$\Gamma_{n_1,\dots,n_{d-1}}(\varphi,\psi) = \int_K dx_1 \cdots dx_{d-1} \int_K dy_1 \cdots dy_{d-1}$$
$$\cdot\, \varphi(x_1,\dots,x_{d-1})\, \cos(n_1(x_1-y_1)) \,\cdots\, \cos(n_{d-1}(x_{d-1}-y_{d-1}))\, \psi(y_1,\dots,y_{d-1}),$$

with the condition $\sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} a_{n_1,\dots,n_{d-1}} < \infty$ replaced by another one, which will be made explicit below (see (C.1)), and under which we can easily check that $\Gamma_K(\varphi,\psi)$ is well defined for each $\varphi,\psi \in C^\infty(K)$.

Let us finally define the covariance Γ_D by

$$\Gamma_D(\varphi,\psi) = \Gamma_K\left(\varphi\big|_K, \psi\big|_K\right), \qquad \varphi,\psi \in \mathcal{S}(D).$$

where $\mathcal{S}(D)$ is the space defined in Chapter 2.

We are going to show that there exists a weak solution to equation (2.1) if and only if the following condition is satisfied:

$$\sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} \frac{a_{n_1,\dots,n_{d-1}}}{\sqrt{1+n_1^2+\cdots+n_{d-1}^2}} < \infty. \tag{C.1}$$

Let us now state the theorem (note that a similar result was already obtained in [19, Thm 13.3.1] for the heat equation).

Theorem C.0. 1. *Let $(u_0,v_0) \in L^2(D) \oplus H^{-1}(D)$. There exists a unique weak solution u of equation (2.1) such that $\mathbb{E}(\|u(t)\|_0^2) < \infty$, for all $t \in \mathbb{R}_+$, if and only if condition (C.1) is satisfied.*

Proof. Let us first compute the eigenvalues and eigenfunctions of the Laplacian in D. Note that since the boundary of the domain is not C^∞, we cannot apply directly the spectral theorem 2.1.1. Still, the solutions of the eigenvalue problem

$$\Delta\varphi + \lambda\varphi = 0 \text{ in } D \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu}\Big|_{\partial D} = 0,$$

are easy to compute here. They have the following simple expressions:

$$e_{\underline{n}}(x) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \cos(n_1x_1) \,\cdots\, \cos(n_dx_d), \qquad \lambda_{\underline{n}} = n_1^2+\cdots+n_d^2,$$

where $\underline{n} = (n_1, \ldots, n_d)$ denotes a multi-index in $\mathbb{N}^d$. One can notice that the $e_{\underline{n}}$ are C^∞, even if ∂D is not.

Let us now compute the coefficients $\gamma_{\underline{n}} = \Gamma_D(e_{\underline{n}}, e_{\underline{n}})$:

$$\gamma_{\underline{n}} = \sum_{m_1,\ldots,m_{d-1}\in\mathbb{N}} a_{m_1,\ldots,m_{d-1}} \Gamma_{m_1,\ldots,m_{d-1}} \left(e_{\underline{n}}\big|_K, e_{\underline{n}}\big|_K\right).$$

Since

$$e_{\underline{n}}\big|_K(x_1, \ldots, x_{d-1}) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \cos(n_1 x_1) \ \cdots \ \cos(n_{d-1} x_{d-1}),$$

and using again the formula $\cos(m(x-y)) = \cos(mx)\cos(my) + \sin(mx)\sin(my)$, we obtain simply that

$$\gamma_{\underline{n}} = \frac{2}{\pi}\, a_{n_1,\ldots,n_{d-1}}.$$

In order to check now that condition (C.1) is sufficient, we simply need to check that it implies that Assumption H_0 of Chapter 2 is satisfied, which in turn implies the desired result by Theorem 2.5.3. To see that part (i) of Assumption H_0 is satisfied, that is, that the covariance Γ_D is continuous with respect to the H^1-norm on D, we follow the proof of Theorem 2.5.3 and verify that Γ_K is continuous with respect to the $H^{\frac{1}{2}}$-norm on K defined by

$$\|\varphi\|_{\frac{1}{2}}^2 = \sum_{n_1,\ldots,n_{d-1}\in\mathbb{N}} \sqrt{1 + n_1^2 + \ldots + n_{d-1}^2}\, |c_{n_1,\ldots,n_{d-1}}|^2,$$

for

$$\varphi(x_1, \ldots, x_{d-1}) = \sum_{n_1,\ldots,n_{d-1}\in\mathbb{N}} c_{n_1,\ldots,n_{d-1}} \ \cos(n_1 x_1) \ \cdots \ \cos(n_{d-1} x_{d-1}).$$

Let us then compute

$$\Gamma_K(\varphi, \varphi) = \sum_{m_1,\ldots,m_{d-1}\in\mathbb{N}} a_{m_1,\ldots,m_{d-1}} \left|c_{m_1,\ldots,m_{d-1}}\right|^2.$$

Under condition (C.1), there exists $C > 0$ such that

$$a_{m_1,\ldots,m_{d-1}} \leq C\, \sqrt{1 + m_1^2 + \ldots + m_{d-1}^2},$$

therefore,

$$\Gamma_K(\varphi, \varphi) \leq C\, \|\varphi\|_{\frac{1}{2}}^2.$$

Let us now verify part (ii) of Assumption H_0 and compute

$$\begin{aligned} \sum_{\underline{n}\in\mathbb{N}^d} \frac{\gamma_{\underline{n}}}{1+\lambda_{\underline{n}}} &= \frac{2}{\pi} \sum_{n_1,\ldots,n_d\in\mathbb{N}} \frac{a_{n_1,\ldots,n_{d-1}}}{1 + n_1^2 + \cdots + n_d^2} \\ &= \frac{2}{\pi} \sum_{n_1,\ldots,n_{d-1}\in\mathbb{N}} a_{n_1,\ldots,n_{d-1}} \left(\sum_{n_d\in\mathbb{N}} \frac{1}{1 + n_1^2 + \cdots + n_d^2} \right). \end{aligned} \tag{C.2}$$

Since

$$\sum_{n_d\in\mathbb{N}} \frac{1}{a^2+n_d^2} \leq \frac{1}{a^2} + \int_0^\infty dx\, \frac{1}{a^2+x^2} = \frac{1}{a^2} + \frac{\pi}{2a} \leq \frac{C_1}{a}, \qquad a \geq 1,$$

we see that (C.1) implies Assumption H_0, therefore the existence of a weak solution u of equation (2.1) such that $\mathbb{E}(\|u(t)\|_0^2) < \infty$, for all $t \in \mathbb{R}_+$, by Theorem 2.5.3. On the other hand, if such a solution exists, then by Theorem 2.5.4, part (ii) of Assumption H_0 is satisfied, and since

$$\sum_{n_d\in\mathbb{N}} \frac{1}{a^2+n_d^2} \geq \int_0^\infty dx\, \frac{1}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right)\Big|_0^\infty = \frac{\pi}{2a} \geq \frac{C_2}{a}$$

we obtain by (C.2) that condition (C.1) is satisfied, which proves that the latter is a necessary condition, therefore the theorem. □

Note that we have obtained here a necessary and sufficient condition, which was not the case for the noise on a sphere. Performing the same analysis as in Remark 3.3.4, we can show that this condition is equivalent to the existence of a trace-class linear operator Q_K on $H^{\frac{1}{2}}(K)$ such that

$$\Gamma_K(\varphi,\psi) = \langle \varphi, Q_K\psi \rangle_{\frac{1}{2}}, \qquad \forall \varphi,\psi \in C^\infty(K).$$

Appendix D

Higher order hyperbolic linear equation in $\mathbb{R}^d$

Let $c \in \mathbb{R}_+$ and let us consider the following fourth order linear partial differential operator:

$$L_c = \frac{\partial^2}{\partial t^2} - 2c\,\frac{\partial \Delta}{\partial t} + \Delta^2$$

What is interesting with this operator is that when $c = 0$, it is given by

$$L_0 = \frac{\partial^2}{\partial t^2} + \Delta^2,$$

which is a truly hyperbolic operator, but when $c = 1$, L is given by

$$L_1 = \frac{\partial^2}{\partial t^2} - 2\frac{\partial \Delta}{\partial t} + \Delta^2 = \left(\frac{\partial}{\partial t} - \Delta\right)^2,$$

which is rather a parabolic operator. Therefore, the analysis will be different in each case.

In the following two sections, we give sufficient conditions which guarantee the existence of a real-valued weak solution u (in the sense defined in Chapter 7) of the following (formal) classical equation:

$$L_c u(t,x) = \dot{F}(t,x),$$

in both cases where $\dot{F}$ is either a spatially homogeneous noise or a noise concentrated on a hyperplane. Since computations are similar to those made in Chapters 7 and 9, we omit (most of) the details in the following.

D.1 Spatially homogeneous noise

Let us consider the following equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - 2c\,\dfrac{\partial \Delta u}{\partial t}(t,x) + \Delta^2 u(t,x) = \dot{F}(t,x), \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\[2ex] u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0, \qquad x \in \mathbb{R}^d, \end{cases} \tag{D.1}$$

where the noise $\dot{F}$ is a spatially homogeneous noise on $\mathbb{R}^d$ with non-negative covariance Γ.

We do not present here what kind of weak formulation can be given to equation (D.1), but we simply give the expression for its distribution-valued solution:

$$\langle u(t), \varphi \rangle = \int_{[0,t]\times\mathbb{R}^d} M(ds, dx)\ (G(t-s) * \varphi)(x), \qquad t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where G is the Green kernel of equation (D.1) and M is the martingale measure constructed from the spatially homogeneous noise $\dot{F}$.

Moreover, we have the following explicit expression for the Fourier transform in x of the Green kernel G of equation (D.1):

$$\mathcal{F}G(t,\xi) = \begin{cases} e^{-c|\xi|^2 t}\, \dfrac{\sin(t\sqrt{1-c^2}\ |\xi|^2)}{\sqrt{1-c^2}\ |\xi|^2}, & \text{if} \quad c \in [0,1[, \\ e^{-c|\xi|^2 t}\ t, & \text{if} \quad c = 1, \\ e^{-c|\xi|^2 t}\, \dfrac{\sinh(t\sqrt{c^2-1}\ |\xi|^2)}{\sqrt{c^2-1}\ |\xi|^2}, & \text{if} \quad c > 1. \end{cases} \tag{D.2}$$

We have the following upper bounds for $\mathcal{F}G$.

Lemma D.1.1. *Let $c = 0$. Then for all $t > 0$, there exists $C(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \le \frac{C(t)}{(1+|\xi|^2)^2}, \qquad \forall \xi \in \mathbb{R}^d.$$

Proof. If $|\xi| \ge 1$, then

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ \frac{\sin^2(s|\xi|^2)}{|\xi|^4} \le \frac{t}{|\xi|^4},$$

and if $|\xi| < 1$, then

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ \frac{\sin^2(s|\xi|^2)}{|\xi|^4} \le t^3,$$

since $r^{-2}\ \sin(r)^2 \le 1$ for all $r \ge 0$. The proof now ends as the proof of Lemma 5.4.1. □

Lemma D.1.2. *Let $c > 0$. Then for all $t > 0$, there exists $C(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \le \frac{C(t)}{(1+|\xi|^2)^3}, \qquad \forall \xi \in \mathbb{R}^d.$$

Proof. Let us first consider the case where $c \in\ ]0,1[$. We have

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ e^{-2c|\xi|^2 s}\, \frac{\sin^2(s\sqrt{1-c^2}\ |\xi|^2)}{(1-c^2)\ |\xi|^4}.$$

If $|\xi| \ge 1$, then

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \le \frac{1}{|\xi|^4}\left(\frac{1-e^{-2c|\xi|^2}t}{2c|\xi|^2}\right) \le \frac{1}{2c|\xi|^6},$$

and if $|\xi| < 1$, then since $r^{-2}\ \sin(r)^2 \leq 1$ for all $r \geq 0$,

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \leq \int_0^t ds\ e^{-2|\xi|^2 s}\ s^2 \leq t^3,$$

For the case $c = 1$, we have, by successive integrations by parts,

$$\begin{aligned}
\int_0^t ds\ \mathcal{F}G(s,\xi)^2 &= \int_0^t ds\ e^{-2|\xi|^2 s}\ s^2 \\
&= -\frac{e^{-2|\xi|^2 t}}{2|\xi|^2}\ t^2 + 2\int_0^t ds\ \frac{e^{-2|\xi|^2 s}}{2|\xi|^2}\ s \\
&= -\frac{e^{-2|\xi|^2 t}}{2c|\xi|^2}\ t^2 + \frac{e^{-2|\xi|^2 t}}{2|\xi|^4}\ t + \frac{1}{4|\xi|^4}\left(\frac{1-e^{-2|\xi|^2 t}}{2|\xi|^2}\right) \\
&\leq \frac{C(t)}{|\xi|^6},
\end{aligned}$$

if $|\xi| \geq 1$, and if $|\xi| < 1$, then

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ e^{-2|\xi|^2 s}\ s^2 \leq t^3.$$

Finally, consider the case $c > 1$. We have

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 = \int_0^t ds\ e^{-2c|\xi|^2 s}\ \frac{\sinh^2(s\sqrt{c^2-1}\ |\xi|^2)}{(c^2-1)\ |\xi|^4}.$$

If $|\xi| \geq 1$, then since $\sinh^2(r) \leq e^{2r}$ for all $r \geq 0$ and $c - \sqrt{c^2-1} > 0$,

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \leq \frac{1}{(c^2-1)\ |\xi|^4}\int_0^t ds\ e^{-2(c-\sqrt{c^2-1})|\xi|^2 s} \leq \frac{1}{(c^2-1)\ 2(c-\sqrt{c^2-1})|\xi|^6}.$$

On the other hand, if $|\xi| < 1$, then since $r^{-2}\ \sinh^2(r) \leq \cosh(r)^2$ for all $r \geq 0$,

$$\int_0^t ds\ \mathcal{F}G(s,\xi)^2 \leq \int_0^t ds\ e^{-2c|\xi|^2 s}\ \cosh^2(s\sqrt{c^2-1}\ |\xi|^2) \leq t,$$

since $\cosh^2(r) \leq e^{2r}$ for all $r \geq 0$ and $\sqrt{c^2-1} \leq c$.

For these three cases, the proof ends as the proof of Lemma 5.4.1. □

We address now the following question: under which condition on the covariance Γ (or equivalently the spectral measure μ) of the noise does there exist a real-valued process X such that

$$\langle u(t), \varphi\rangle = \int_{\mathbb{R}^d} dx\ X(t,x)\ \varphi(x), \qquad \mathbb{P}-a.s.,\ \forall t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d)?$$

The answer to this question is given in the following theorem.

Theorem D.1.3. *If*

$$c = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^2} < \infty, \tag{D.3}$$

or

$$c > 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^3} < \infty, \tag{D.4}$$

then there exists a real-valued process X which is the weak solution of equation (D.1).

Proof. We can use here the result in [15], which states that there exists a real-valued process which is the weak solution of (D.1) if the following condition on μ is satisfied:

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^T ds\ \mathcal{F}G(s,\xi)^2 < \infty,$$

for all $T > 0$. The theorem then follows directly from Lemmas D.1.1 and D.1.2. □

Note that when μ is the Lebesgue measure on $\mathbb{R}^d$ (that is, the spectral measure of white noise), condition (D.3) is satisfied if and only if $d < 4$ and condition (D.4) is satisfied if and only if $d < 6$.

D.2 Noise on a hyperplane

We now consider the following equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - 2c\, \dfrac{\partial \Delta u}{\partial t}(t,x) + \Delta^2 u(t,x) = \dot{F}(t,x_1)\ \delta_0(x_2), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = \dfrac{\partial u}{\partial t}(0,x) = 0, \qquad x \in \mathbb{R}^d, & \end{cases} \tag{D.5}$$

where $\dot{F}$ is the noise concentrated on the hyperplane $x_2 = 0$ considered in Chapter 6 (with $k = d-1$), with non-negative covariance Γ.

As in the preceding section, we do not present here what kind of weak formulation can be given to equation (D.5), but we simply give the expression for its distribution-valued solution:

$$\langle u(t), \varphi\rangle = \int_{[0,t]\times\mathbb{R}^{d-1}} M(ds,dx_1)\ (G(t-s) * \varphi)(x_1,0), \qquad t \in \mathbb{R}_+,\ \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where G is the Green kernel of equation (D.1) and M is the martingale measure constructed from the noise $\dot{F}$.

The expression of $\mathcal{F}_1 G$ is given by (4.1):

$$\mathcal{F}_1 G(t,\xi_1,x_2) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi_2\ \mathcal{F}G(t,\xi_1,\xi_2)\ \chi_{-x_2}(\xi_2), \tag{D.6}$$

and we will not compute it since its explicit expression is too intricate. On the other hand, we have the two following upper bounds. We restrict ourselves to the two cases where either $c = 0$ or $c = 1$ for simplicity.

Lemma D.2.1. *Let $c = 0$. Then for all $t > 0$, there exists $C(t) > 0$ such that*

$$\int_0^t ds\ \mathcal{F}_1 G(s,\xi_1,x_2)^2 \le \frac{C(t)}{1+|\xi_1|^2} \qquad \forall(\xi_1,x_2) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

Proof. We have

$$\mathcal{F}_1 G(t,\xi_1,x_2) = \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2 \, \frac{\sin((|\xi_1|^2+\xi_2^2)s)}{|\xi_1|^2+\xi_2^2}\, e^{-ix_2\xi_2},$$

therefore, when $|\xi| \geq 1$,

$$|\mathcal{F}_1 G(t,\xi_1,x_2)| \leq \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2 \, \frac{1}{|\xi_1|^2+\xi_2^2} = \frac{1}{2|\xi_1|},$$

so

$$\int_0^t ds \; \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{t}{4|\xi_1|^2}.$$

When $|\xi_1| < 1$, we have, since $r^{-2}\,\sin(r)^2 \leq C(1+r^2)^{-1}$ for all $r \geq 0$,

$$|\mathcal{F}_1 G(t,\xi_1,x_2)| \leq \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2 \, \frac{C}{1+\xi_2^2} = \frac{C}{2},$$

therefore,

$$\int_0^t ds \; \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{C^2 t}{4},$$

and the proof ends as the proof of Lemma 5.4.1. □

Lemma D.2.2. *Let $c = 1$. Then for all $t > 0$, there exists $C(t) > 0$ such that*

$$\int_0^t ds \; \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{C(t)}{(1+|\xi_1|^2)^2} \qquad \forall (\xi_1,x_2) \in \mathbb{R}^{d-1}\times\mathbb{R}.$$

Proof. When $c = 1$, we can compute $\mathcal{F}_1 G$ explicitely:

$$\begin{aligned}
\mathcal{F}_1 G(t,\xi_1,x_2) &= \frac{1}{2\pi}\int_{\mathbb{R}} d\xi_2 \; e^{-(|\xi_1|^2+\xi_2^2)t}\; t\; e^{-ix_2\xi_2} \\
&= e^{-|\xi_1|^2 t}\; t \int_{\mathbb{R}} d\xi_2 \; e^{-\xi_2^2 t}\; e^{-ix_2\xi_2} \\
&= e^{-|\xi_1|^2 t}\sqrt{\frac{t}{4\pi}}\; e^{-\frac{x_2^2}{4t}}.
\end{aligned} \tag{D.7}$$

We therefore obtain, for all $x_2 \in \mathbb{R}$,

$$\int_0^t ds \; \mathcal{F}_1 G(s,\xi_1,x_2)^2 = \frac{1}{4\pi}\int_0^t ds \; e^{-2|\xi_1|^2 s}\; s\; e^{-\frac{x_2^2}{2s}} \leq \frac{1}{4\pi}\int_0^t ds \; e^{-2|\xi_1|^2 s}\; s.$$

By integration by parts, we have

$$\int_0^t ds \; \mathcal{F}_1 G(s,\xi_1,x_2)^2 = \frac{e^{-2|\xi_1|^2 t}}{8\pi|\xi_1|^2} + \frac{1-e^{-2|\xi_1|^2 t}}{16\pi|\xi_1|^4} \leq \frac{C(t)}{|\xi_1|^4},$$

when $|\xi_1| \geq 1$, and when $|\xi_1| < 1$, we have

$$\int_0^t ds \; \mathcal{F}_1 G(s,\xi_1,x_2)^2 \leq \frac{1}{4\pi}\int_0^t ds \; e^{-2|\xi_1|^2 s}\; s \leq \frac{t^2}{4\pi}.$$

The proof now ends as the proof of Lemma 5.4.1. □

The following upper bound shows moreover the analogy between equation (D.1) with $c = 1$ and the heat equation (9.1).

Lemma D.2.3. *Let $c = 1$. Then for all $T > 0$ and $\varepsilon > 0$, there exist $C(T, \varepsilon) > 0$ and P a function with polynomial growth such that*

$$\int_0^t ds\ \mathcal{F}_1 G(s, \xi_1, x_2)^2 \leq C(T, \varepsilon)\ P(\xi_1)\ e^{-2\,\varepsilon\,|\xi_1|},$$

for all $t \in [0, T]$, $\xi_1 \in \mathbb{R}^{d-1}$ and $x_2 \in \mathbb{R}$ such that $|x_2| \geq \varepsilon$.

Proof. From the explicit expression (D.7) of $\mathcal{F}_1 G$ and following an argument entirely similar to that of the proof of Lemma 9.2.1, we obtain the result. □

These estimates lead to the following theorem.

Theorem D.2.4. *If*

$$c = 0 \quad and \quad \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{1 + |\xi_1|^2} < \infty, \tag{D.8}$$

or

$$c = 1 \quad and \quad \int_{\mathbb{R}^{d-1}} \frac{\mu(d\xi_1)}{(1 + |\xi_1|^2)^2} < \infty, \tag{D.9}$$

then there exists a real-valued process X which is solution of equation (D.5). Moreover, if $c = 1$, there always exists a real-valued process defined outside the hyperplane $x_2 = 0$ which is solution of equation (D.5), without any specific assumption on μ.

Proof. The proof follows the same scheme as the proof for the second order hyperbolic equation in Chapter 7. We will therefore not go into the details of this proof. Just note that Lemmas D.2.1, D.2.2 and D.2.3 are the essential ingredients of the argument (as was Lemma 7.1.2 for Theorem 7.2.5). □

Note that when μ is the Lebesgue measure on $\mathbb{R}^{d-1}$ (that is, the spectral measure of white noise), condition (D.8) is satisfied if and only if $d < 3$ (3 being a limiting case) and condition (D.9) is satisfied if and only if $d < 4$.

For a further analysis, we could also study the case $c \neq 0, 1$ in this section (for which the answer is not clear a priori) and see if all the conditions obtained in this appendix are optimal.

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Curriculum Vitae

I was born on October 2nd 1971 in Geneva, Switzerland. After my primary school and first years of secondary school in Collonge-Bellerive, a nice little village at the lake side near Geneva, I attended the *Collège Calvin* in Geneva, where I obtained the *Maturité Fédérale et Cantonale de type A (grec-latin)* in June 1990. I then studied physics at the Swiss Federal Institute of Technology (EPFL) and graduated with a *Diplôme d'ingénieur-physicien* in March 1995. After a trainee period of three months at the Physics Department of the EPFL, I became a research and teaching assistant in October 1995 at the Mathematics Department of the EPFL, where I first worked in Numerical Analysis with Prof. Jacques Rappaz until August 1996, then in Probability Theory with Prof. Robert C. Dalang. I began to work on the present PhD dissertation in October 1997 under the supervision of Professor Robert C. Dalang. During this period, I also got married with Marivi on May 1st 1998.