Scaling Laws for One-dimensional Ad Hoc Wireless Networks

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Abstract— **We obtain a precise information theoretic upper bound on the rate per communication pair in a one-dimensional ad hoc wireless network. The key ingredient of our result is a uniform upper bound on the determinant of the Cauchy matrix.**

I. Introduction

[1] The study of the capacity of ad hoc wireless networks goes back to the seminal paper of P. Gupta and P. R. Kumar [1], in which they prove, under some realistic assumptions regarding state of the art wireless communications, that the transport capacity of planar ad hoc networks grows asymptotically at most like the square root of the number of users in the network. One still misses a confirmation of this result from an information theoretic point of view (i.e., without any assumption on the way communications are established in the network). Some attempts have been performed recently (see for instance [2]–[4]), all leading to partial answers.

The argument of Gupta and Kumar can be easily translated to one-dimensional networks and leads to the conlusion that in this case, the transport capacity of the network grows asymptotically at most like the number of users. Even though the analysis is (much) simpler in the one-dimensional case, no complete information theoretic confirmation of this result has been given so far.

In [3], P. R. Kumar and L.-L. Xie consider an arbitrary one-dimensional network composed of n users separated by a minimum distance $d > 0$ and show that the transport capacity of such networks does not grow faster than n, provided that the attenuation function of the transmitted signals over distance is given by

$$g(r) = \frac{e^{-\beta r/2}}{r^{\alpha/2}},$$

where either $\beta > 0$ or $\alpha > 4$ (note that g describes the decay of the amplitude of the electric field and not that of the power).

In the particular scenario where order n pairs chosen at random wish to establish communication, the above result implies that the maximum achievable rate R per communication pair decreases like $\frac{1}{n}$ as n gets large, since order n communications need to be established over distances of order n on average (recall that the network is assumed to be one-dimensional and that users are separated by a minimum distance d).

In the case $\beta = 0$, the above assumption that $\alpha > 4$ is quite unrealistic regarding wireless communications. In [4], O. Lévêque and E. Telatar consider uniformly distributed one-dimensional networks (with a constant density of users) and show that in the above mentioned scenario, if $\alpha > 1$, then

$$R \leq K \, \frac{\log n}{n^{1-\frac{1}{\alpha}}}, \tag{1}$$

so that R tends to zero as n gets large. However, this upper bound is as not tight as the $\frac{1}{n}$ behaviour obtained under a weaker assumption on α in [3]. In the following, we consider an arbitrary one-dimensional network with users separated by a minimum distance $d > 0$ and show (see theorem 3.1) that if $2 \leq \alpha \leq 4$, then the following upper bound holds true:

$$\forall \varepsilon > 0, \quad \exists K > 0 \quad \text{such that} \quad R \leq K \, \frac{(\log n)^{3+\varepsilon}}{n}, \tag{2}$$

which is tighter than (1), especially for small α. With this respect, our result below closes the information theoretic gap left open in [3] concerning one-dimensional ad hoc networks.

Note finally that a result similar in spirit has been obtained by A. Jovicic et al. in [2]: for one-dimensional networks and under the slightly different propagation model

$$\tilde{g}(r) = \frac{e^{i\phi}}{(1+r)^{\alpha/2}},$$

where ϕ is a random phase, they prove that the transport capacity does not grow faster than n, provided that $\alpha > 3$ and that the users have a perfect knowledge of the phases, or provided that $\alpha > 2$ but that users have no information about the phases.

II. Our Approach

We follow here the lines of [4], specializing the model to arbitrary one-dimensional networks.

We consider a network of n users (with n even for simplicity) arbitrarily placed on the real line, but separated by a minimum distance $d > 0$. Among these n users, we choose $n/2$ users at random and assume that each of these users wishes to establish communication with a correspondent chosen at random in the other group of $n/2$ users (without

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any consideration on their respective locations). We assume that there is no fixed infrastructure that helps relaying communications, but we also assume no restriction on the kind of help the users can give to each other; in particular, any user may act as a relay for the communicating pairs, but we may also imagine more sophisticated group communications and interference cancellation strategies. We further assume that in order to establish communication, each user has a device of power P. The attenuation of the transmitted signals over distance is governed by the function $g(r)$ given by $g(r) = \frac{1}{r^{\alpha/2}}$ with $2 \leq \alpha \leq 4$. For notational convenience, let us define the coefficient $\delta = \alpha/2$ (corresponding to the coefficient δ defined in [3]).

We divide the network into two parts, so that there are exactly $n/2$ users on each side, and place the origin at the middle point between the two most "central" users. There are therefore $n/2$ users located left to the origin; statistically, half of these are transmitters and half of these transmitters wish to establish communication with a receiver located right to the origin. In total, there are therefore about $n/8$ communications which need to be carried over the origin from left to right, and deviations from this idealized situation are of order much smaller than n with high probability.

Let R be the the maximum achievable rate per communication pair in the network. In order to obtain an upper bound on R, we first assume that only the above $n/8 + o(n)$ communications need to be established. We then introduce n additional "mirror" user that help relaying communications (where the mirror location of $x \in \mathbb{R}$ is $\tilde{x} = -x$). There are now n users on each side of the origin whose positions are denoted respectively by

$$x_1, \ldots, x_n \quad \text{and} \quad y_1, \ldots, y_n, \quad \text{with } y_i = -x_i.$$

Without any restriction of generality, we may order the points so that $x_1 \leq \ldots \leq x_n$. By the constraint imposed on the minimum distance, we obtain that $x_1 \geq \frac{d}{2}$ and that $x_i \geq (i-1)\,\frac{d}{2}$ for all $i \in \{2, \ldots, n\}$.

Using the classical cut-set bound of [5, Thm 14.10.1] and following the argument of [4], we obtain that

$$R \leq \frac{C_n}{n/8 + o(n)}, \tag{3}$$

where C_n is the capacity of the vector channel given by

$$Y_j = \sum_{i=1}^{n} G_{ij}^{(\delta)}\, X_i + Z_j, \quad j = 1, \ldots, n,$$

with

$$G_{ij}^{(\delta)} = \frac{1}{|x_i - y_j|^\delta} = \frac{1}{(x_i + x_j)^\delta}$$

and $Z = (Z_1, \ldots, Z_n)$ is a vector of independent circularly symmetric complex Gaussian random variables with unit variance. Under the power constraint

$$\sum_{i=1}^{n} \mathbb{E}(|X_i|^2) \leq nP,$$

the capacity of the above channel is given by

$$C_n = \max_{P_k \geq 0 : \sum_{k=1}^n P_k \leq nP} \sum_{k=1}^{n} \log(1 + P_k\, \lambda_k^2), \tag{4}$$

where λ_k are the eigenvalues of the symmetric matrix $G^{(\delta)}$. Noting that $P_k \leq nP$ for each k and that the λ_k are nonnegative (see [4]), we further obtain that

$$\begin{aligned} C_n &\leq \sum_{k=1}^{n} \log(1 + nP\, \lambda_k^2) \\ &\leq 2 \sum_{k=1}^{n} \log(1 + \sqrt{nP}\, \lambda_k) \\ &= 2\, \log\det(I + \sqrt{nP}\, G^{(\delta)}). \end{aligned} \tag{5}$$

III. Main Result

Theorem 3.1: Let $1 \leq \delta \leq 2$ (or equivalently, $2 \leq \alpha \leq 4$). For all $\varepsilon > 0$, there exists a constant $K > 0$ (independent of n and δ) such that the capacity C_n is bounded above by

$$C_n \leq K\,(\log n)^{3+\varepsilon}, \quad \text{for sufficiently large } n,$$

so this estimate combined with (3) implies the asymptotic upper bound (2) on the maximum achievable rate R per communication pair in the network.

The remainder of the section is devoted to the proof of the above theorem. Using (5) and the following classical identity, valid for any $n \times n$ matrix A:

$$\det(I + A) = \sum_{J \subset \{1,\ldots,n\}} \det(A(J)), \text{ where } A(J) = (a_{ij})_{i,j \in J}, \tag{6}$$

we obtain that

$$\begin{aligned} \exp(C_n/2) &\leq \det(I + \sqrt{nP}\, G^{(\delta)}) \\ &= \sum_{m=0}^{n} (nP)^{m/2} \sum_{\substack{J \subset \{1,\ldots,n\} \\ |J| = m}} \det(G^{(\delta)}(J)). \end{aligned} \tag{7}$$

Let us introduce the notation:

$$D_\delta(\mathbf{x}_J) = \det\left(\left(\frac{1}{(x_i + x_j)^\delta}\right)_{i,j \in J}\right) = \det(G^{(\delta)}(J)), \tag{8}$$

where $\mathbf{x}_J = (x_i)_{i \in J}$. We now give a series of lemmas concerning $D_\delta(\mathbf{x}_J)$ and related sketchs of proofs.

Lemma 3.2: Let $\delta > 0$ and $J \subset \{1, \ldots, n\}$. The determinant defined in (8) satisfies the following identity:

$$D_\delta(\mathbf{x}_J) = \frac{1}{m!\,\Gamma(\delta)^m} \left(\prod_{i \in J} \int_{\mathbb{R}_+} dt_i\, t_i^{\delta - 1}\right) \det\left(\left(e^{-t_i\, x_j}\right)_{i,j \in J}\right)^2,$$

where $m = |J|$ and Γ is the Euler Gamma function.

Proof: The above lemma follows from the fact that

$$\frac{1}{(x_i + x_j)^\delta} = \frac{1}{\Gamma(\delta)} \int_{\mathbb{R}_+} dt\, t^{\delta-1}\, e^{-t\,(x_i + x_j)}$$

and the following two equivalent definitions of the determinant of a matrix:

$$\begin{aligned}\det(A(J)) &= \sum_{\sigma\in\mathcal{S}(J)} \varepsilon(\sigma)\prod_{i\in J} a_{i,\sigma(i)} \\ &= \frac{1}{m!}\sum_{\sigma,\tau\in\mathcal{S}(J)} \varepsilon(\sigma)\,\varepsilon(\tau)\prod_{i\in J} a_{\sigma(i),\tau(i)}, \quad (9)\end{aligned}$$

where $\mathcal{S}(J)$ is the set of permutations of J and $\varepsilon(\sigma)$ is the signature of the permutation σ. ■

Lemma 3.3: For all $J \subset \{1,\ldots,n\}$ and $1 \le \delta \le 2$,

$$D_\delta(\mathbf{x}_J) \le \frac{1}{\Gamma(\delta)^m}\, D_2(\mathbf{x}_J)^{\delta-1}\, D_1(\mathbf{x}_J)^{2-\delta},$$

where $m = |J|$.

Proof: The above interpolation formula follows from the identity obtained in lemma 3.2 and the following consequence of Hölder's inequality, valid for any $1 \le \delta \le 2$ and $f : \mathbb{R}_+ \to \mathbb{R}$ measurable such that both $f(t)^2$ and $t\,f(t)^2$ are integrable on $\mathbb{R}_+$:

$$\int_{\mathbb{R}_+} dt\, t^{\delta-1}\, f(t)^2 \le \left(\int_{\mathbb{R}_+} dt\, t\, f(t)^2\right)^{\delta-1} \left(\int_{\mathbb{R}_+} dt\, f(t)^2\right)^{2-\delta}.$$

■

Let us now recall the definition of the *permanent* of a $m\times m$ matrix $A(J) = (a_{ij})_{i,j\in J}$:

$$\mathrm{perm}(A(J)) = \sum_{\sigma\in\mathcal{S}(J)}\prod_{i\in J} a_{i,\sigma(i)}.$$

We also define for $\delta > 0$ and $J \subset \{1,\ldots,n\}$:

$$P_\delta(\mathbf{x}_J) = \mathrm{perm}\left(\left(\frac{1}{(x_i+x_j)^\delta}\right)_{i,j\in J}\right).$$

The following identity is due to Borchardt (see for instance [6]):

$$D_2(\mathbf{x}_J) = D_1(\mathbf{x}_J)\, P_1(\mathbf{x}_J). \quad (10)$$

Combining this identity with lemma 3.3, we obtain that there exists $K > 0$ such that

$$D_\delta(\mathbf{x}_J) \le K^m\, D_1(\mathbf{x}_J)\, P_1(\mathbf{x}_J)^{\delta-1}. \quad (11)$$

Given the constraints on the positions x_i, that is, $x_1 \ge \frac{d}{2}$ and $x_i \ge (i-1)\,\frac{d}{2}$, it is easily seen that

$$P_1(\mathbf{x}_J) \le m\,\left(\frac{2}{d}\right)^m. \quad (12)$$

We further obtain the following key estimate on $D_1(\mathbf{x}_J)$.

Lemma 3.4: Under the only assumption that $x_n \ge \ldots \ge x_1 \ge \frac{d}{2}$ (and that $|J| = m$), there exists $K > 0$ such that

$$D_1(\mathbf{x}_J) \le \left(\frac{2}{d}\right)^m \exp(-K\, m^{3/2}). \quad (13)$$

Proof: The proof of the above inequality is rather long, so we only give here a sketch idea (for a more detailed exposition, the reader is referred to [7]). We have the following nice analytic expression for $D_1(\mathbf{x}_J)$, due to Cauchy (see for instance [8, p. 202]):

$$D_1(\mathbf{x}_J) = \left(\prod_{\substack{i,j\in J\\ i<j}} \frac{x_j - x_i}{x_i + x_j}\right)^2 \prod_{i\in J}\frac{1}{2x_i}. \quad (14)$$

The constraints on the x_i are independent of the set J considered, so we may as well assume that $J = \{1,\ldots,m\}$. Let us further assume that the vector $\mathbf{x} = (x_i)_{i=1}^m$ is such that D_1 reaches its maximum at $\mathbf{x}$ (notice that $0 \le D_1(\mathbf{x}) \le \prod_{1\le i\le m}\frac{1}{2x_i}$, so that the vector $\mathbf{x}$ exists). It is therefore true that for all $k \in \{2,\ldots,m-1\}$, the function g_k defined as

$$g_k(t) = -\log D_1(x_1,\ldots,x_k,tx_{k+1},\ldots,tx_m)$$

is minimum in $t = 1$, so that $g_k'(1) = 0$. Computing explicitly $g_k'(1)$ leads us (again, see [7] for more details) to the conclusion that if the vector $\mathbf{x}$ maximizes D_1, then there exists $K > 0$ such that

$$x_i \ge K\,\frac{d}{2}\,\exp\left(\frac{i-1}{\sqrt{m}}\right), \quad \forall i \in \{1,\ldots,m\}.$$

This allows us to conclude that there exists $\tilde{K} > 0$ such that

$$D_1(\mathbf{x}) \le \prod_{1\le i\le m}\frac{1}{2x_i} \le \exp(-\tilde{K}\, m^{3/2}).$$

■

Finally, combining equations (7), (11), (12) and (13) leads us to

$$\begin{aligned}\exp(C_n/2) &\le \sum_{m=0}^n (nP)^{m/2}\, m\,\left(\frac{4}{d}\right)^m \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=m}} D_1(\mathbf{x}_J) \\ &\le \sum_{m=0}^n (nP)^{m/2}\, m\,\left(\frac{4}{d}\right)^m \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=m}} \left(\frac{2}{d}\right)^m \exp(-K\, m^{3/2}).\end{aligned}$$

Since

$$\sum_{J\subset\{1,\ldots,n\}:|J|=m} 1 = \binom{n}{m} \le n^m,$$

we have

$$\begin{aligned}\exp(C_n/2) &\le \sum_{m=0}^n P^{m/2}\, n^{3m/2}\, m\,\left(\frac{8}{d^2}\right)^m \exp(-K\, m^{3/2}) \\ &\le \sum_{m=0}^n \exp(Lm\log n - Km^{3/2}),\end{aligned}$$

where L is some positive constant. This leads us finally to the fact that there exists $M > 0$ such that

$$\exp(C_n/2) \le \exp(M(\log n)^{3+\varepsilon}),$$

which concludes the proof of theorem 3.1. □

IV. Conclusion

As already mentioned in the introduction, theorem 3.1 closes the information theoretic gap left open in [3] concerning one-dimensional ad hoc networks. Similar analysis techniques are to be developed for the two-dimensional case.

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