On the MIMO Channel Capacity for the Dual and Asymptotic Cases over Hoyt Channels

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***Abstract*— This paper presents the dual (2×2) MIMO channel capacity over the Hoyt fading channel. The joint eigenvalue distribution of the channel matrix is obtained and it is shown that the effect of the Hoyt parameter b can degrade around 8.5% the channel capacity. For the $t\times r$ case, an asymptotic result is also derived. All the results are validated by numerical Monte Carlo simulations and are in excellent agreement.**

***Index Terms*— Fading distributions, Rayleigh distribution, Hoyt distribution, Eigenvalue distribution, MIMO channels.**

I. Introduction

The Shannon capacity of a MIMO (multiple input multiple output) channel can be computed by means of the joint eigenvalue distribution of the matrix $\mathbf{W}=\mathbf{H}^{\dagger}\mathbf{H}$, where $\mathbf{H}$ is the channel matrix and † denotes the complex conjugate transpose. In the classical Rayleigh fading model, the entries of $\mathbf{H}$ are assumed to be i.i.d. (independent and identically distributed) zero mean complex Gaussian with independent real and imaginary parts sharing the same variance. For this model, the joint eigenvalue distribution is known and a closed form expression for the capacity has been obtained in [1]. Unfortunately, as one departs from this standard model, much less is known on the joint eigenvalue distribution. In his pioneering work [2], James obtained a formula for the non-zero mean (or Ricean) case. Based on this work, the capacity of the Ricean MIMO channel was computed in [3], [4].

In the present work, our aim is to address the case where the real and imaginary parts of the entries have different variances, their modulus being therefore distributed according to the Hoyt distribution [5]. The Hoyt distribution [5] spans the range from the one-sided (real) Gaussian distribution to the Rayleigh distribution. It has found applications in the error performance evaluation of digital communication, outage analysis in cellular mobile radio system, or satellite channel modelling [6]. Despite its practical interest, very little attention has been paid to this type of fading. In this paper, a new Hoyt-Wishart distribution is presented for the 2×2 case. Besides, the joint eigenvalue distribution and the channel capacity are computed analytically.

The same approach can be extended to the general $t\times r$ case, but the mathematical complexity becomes prohibitive. In this case, an asymptotic result has been found. Although this formula is asymptotic, it is shown by simulation that it is quite accurate, even for a small number of antennas. Simulations results are used to validate the finite and asymptotic results.

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II. The Hoyt Distribution

The Hoyt fading signal is modeled as $Z=X+j\,Y$ where $j=\sqrt{-1}$, X and Y are independent zero-mean Gaussian random variables with variances σ_X^2 and σ_Y^2, respectively. The probability density function (PDF) of $R=|Z|$ is given by [5]

$$p(r)=\frac{2r}{\Omega\sqrt{1-b^2}}\exp\left(-\frac{r^2}{\Omega(1-b^2)}\right)I_0\left(\frac{b\,r^2}{\Omega(1-b^2)}\right) \quad (1)$$

where $\Omega=\mathrm{E}[R^2]$[1], $b=(\sigma_X^2-\sigma_Y^2)/(\sigma_X^2+\sigma_Y^2)\in[-1,1]$ is the Hoyt parameter, and $I_0(\cdot)$ is the modified Bessel function of the first kind and zeroth order.

III. System Model

We consider a single-user Gaussian channel with t antennas at the transmitter and r antennas at the receiver and refer to it as a $t\times r$ MIMO channel. This channel can be modeled as $\mathbf{y}=\mathbf{H}\,\mathbf{x}+\mathbf{n}$, where $\mathbf{H}$ can be written as

$$\mathbf{H}=\mathbf{H_R}+j\,\mathbf{H_I} \quad (2)$$

and $\mathbf{H_R}$, $\mathbf{H_I}$ are independent $r\times t$ real matrices with i.i.d. zero mean Gaussian entries with variances σ_X^2, σ_Y^2 respectively. Note that $\mathbf{H}$ has i.i.d. complex Gaussian entries with unequal real and imaginary parts. The vector $\mathbf{y}\in\mathcal{C}^r$, $\mathbf{x}\in\mathcal{C}^t$, and $\mathbf{n}$ is zero-mean complex Gaussian noise with $E[\mathbf{n}\,\mathbf{n}^{\dagger}]=\mathbf{I_r}$. The total power of the transmitter is constrained by $E[\mathbf{x}^{\dagger}\mathbf{x}]\le P$.

IV. Channel Capacity

The ergodic channel capacity is given by [1]

$$C=\sup_{\mathbf{Q}\ge0:\mathrm{tr}[\mathbf{Q}]\le P} E\left[\log_2\det\left(\mathbf{I}+\mathbf{HQH}^{\dagger}\right)\right] \quad (3)$$

where $\mathbf{Q}$ is the covariance matrix of $\mathbf{x}$. Here, the entries $\mathbf{H}_{ij}$ are i.i.d. and the distribution of $\mathbf{H}_{ij}$ is the same as $-\mathbf{H}_{ij}$ for all i,j, so we deduce from Corollary 1b in [7] that

$$C=E\left[\log_2\det\left(\mathbf{I}+\frac{P}{t}\mathbf{H}\,\mathbf{H}^{\dagger}\right)\right] \quad (4)$$

Defining then

$$\mathbf{W}=\begin{cases}\mathbf{HH}^{\dagger} & r<t\\ \mathbf{H}^{\dagger}\mathbf{H} & r\ge t\end{cases} \quad (5)$$

$n=\max\{r,t\}$ and $m=\min\{r,t\}$, the capacity can be written in terms of the eigenvalues $\lambda_1,\dots\lambda_m$ of $\mathbf{W}$ as

$$\begin{aligned} C &= E\left[\sum_{i=1}^{m}\log_2\left(1+(P/t)\,\lambda_i\right)\right] && (6)\\ &= m\,E\left[\log_2\left(1+(P/t)\,\lambda\right)\right] && (7)\end{aligned}$$

[1] $E\,[\cdot]$ denotes the expectation operator.

It is worth noticing that the distribution of $\mathbf{W}$, when the variances σ_X^2 and σ_Y^2 of the real and imaginary part of $\mathbf{H}$ are the same, is commonly referred to as the Wishart distribution. In our case, we have a new distribution which we will refer to as the Hoyt-Wishart distribution.

V. The Hoyt 2×2 case

The aim of this section is to compute the joint eigenvalue distribution for the Hoyt-Wishart distribution. Consider the complex 2×2 channel matrix $\mathbf{H}$ given by

$$\mathbf{H} = \mathbf{H}_R + j\mathbf{H}_I \tag{8}$$

where the entries of $\mathbf{H}_R$ are zero mean Gaussian variates with variance σ_X^2 and the entries of $\mathbf{H}_I$ are zero mean Gaussian variates with variance σ_Y^2.

The joint eigenvalue distribution of $\mathbf{W}$ will be found using the following steps: 1) the joint distribution of the entries of $\mathbf{H}$ is the multivariate Gaussian distribution; 2) we then write $\mathbf{H} = \mathbf{QL}$ where $\mathbf{L}$ is an upper triangular matrix and $\mathbf{Q}$ is an unitary matrix $\left(\mathbf{QQ}^\dagger = \mathbf{I}\right)$; 3) therefore, $\mathbf{W} = \mathbf{L}^\dagger\mathbf{L}$; 4) finally, performing the eigendecomposition $\mathbf{W} = \mathbf{S\Lambda S}^\dagger$, it is possible to get the joint eigenvalue distribution.

The distribution of $\mathbf{H}$ can be written as

$$p\left(\mathbf{H}\right) = K\, e^{-\frac{1}{8\sigma_X^2}\mathrm{tr}\left[\left(\mathbf{H}+\mathbf{H}^*\right)\left(\mathbf{H}^T+\mathbf{H}^\dagger\right)\right]+\frac{1}{8\sigma_Y^2}\mathrm{tr}\left[\left(\mathbf{H}-\mathbf{H}^*\right)\left(\mathbf{H}^T-\mathbf{H}^\dagger\right)\right]} \tag{9}$$

where $K = \left(\left(\sqrt{2\pi}\sigma_X\right)^4\left(\sqrt{2\pi}\sigma_Y\right)^4\right)^{-1}$, $\mathrm{tr}\left[\cdot\right]$ denotes the matrix trace, $\left(\cdot\right)^T$ denotes transpose and $\left(\cdot\right)^*$ denotes complex conjugate. Now the following transformation $\mathbf{H} = \mathbf{QL}$ can be performed, where [8]

$$\mathbf{Q} = \left(\begin{array}{ll} e^{j\phi_1}\cos\left(\theta\right) & -e^{j\left(\phi_3-\phi_2\right)}\sin\left(\theta\right) \\ e^{j\phi_2}\sin\left(\theta\right) & e^{j\left(\phi_3-\phi_1\right)}\cos\left(\theta\right) \end{array}\right) \tag{10}$$

$0 \le \phi_1, \phi_2, \phi_3 \le 2\pi$, $0 \le \theta \le \pi/2$, and

$$\mathbf{L} = \left(\begin{array}{ll} l_{11} & l_{12R} + j\, l_{12I} \\ 0 & l_{22} \end{array}\right) \tag{11}$$

The Jacobian of this transformation will be given by $J = l_{11}^3 l_{22}\, \sin\left(\theta\right)\cos\left(\theta\right)$, so carrying out some simplifications, the joint PDF of $\mathbf{Q}$ and $\mathbf{L}$ will be given by

$$\begin{aligned} p\left(\mathbf{Q},\mathbf{L}\right) &= K\, l_{11}^3 l_{22}\, \sin\left(\theta\right)\cos\left(\theta\right) \\ &\times e^{-\mathrm{tr}\left[\mathbf{LL}^\dagger\right]\left(\frac{1}{4\sigma_X^2}+\frac{1}{4\sigma_Y^2}\right)+\mathrm{Re}\left[\mathrm{tr}\left[\mathbf{QLL}^T\mathbf{Q}^T\right]\right]\left(\frac{1}{4\sigma_Y^2}-\frac{1}{4\sigma_X^2}\right)} \end{aligned} \tag{12}$$

where $\mathrm{Re}\left[\cdot\right]$ denotes the real part of a complex quantity. Now we can perform the integration over $\mathbf{Q}^2$ and get $p(\mathbf{L})$. We then perform another transformation of variables

$$\mathbf{W} = \mathbf{L}^\dagger\mathbf{L} = \left(\begin{array}{ll} w_1 & w_3 + jw_4 \\ w_3 - jw_4 & w_2 \end{array}\right)$$

The Jacobian of this transformation is given by $J = 4l_{11}^3 l_{22}$, and we get the 2×2 Hoyt-Wishart distribution given in (13).

[2]We introduce the notation $\int\limits_{\mathcal{Q}} d\mathbf{Q} = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} d\theta d\phi_1 d\phi_2 d\phi_3$

where

$$\begin{aligned} f_1\left(\theta,\phi_1,\phi_2\right) &= \cos\left(\theta\right)^2\cos\left(2\phi_1\right) + \sin\left(\theta\right)^2\cos\left(2\phi_2\right) \\ f_2\left(\theta,\phi_1,\phi_2\right) &= \cos\left(\theta\right)^2\sin\left(2\phi_1\right) + \sin\left(\theta\right)^2\sin\left(2\phi_2\right) \\ f_3\left(\theta,\phi_1,\phi_2,\phi_3\right) &= \cos\left(\theta\right)^2\cos\left(2\left(\phi_1-\phi_3\right)\right) \\ &\quad + \sin\left(\theta\right)^2\cos\left(2\left(\phi_2-\phi_3\right)\right)) \\ f_4\left(\theta,\phi_1,\phi_2\right) &= \sin\left(2\theta\right)\sin\left(\phi_1-\phi_2\right) \end{aligned}$$

Finally, to get the joint eigenvalue distribution, we perform another decomposition $\mathbf{W} = \mathbf{S\Lambda S}^\dagger$ where $\mathbf{S}$ is given by

$$\mathbf{S} = \left(\begin{array}{ll} \cos\left(\kappa\right) & -e^{i\psi}\sin\left(\kappa\right) \\ e^{-i\psi}\sin\left(\kappa\right) & \cos\left(\kappa\right) \end{array}\right). \tag{14}$$

$0 \le \psi \le 2\pi$, $0 \le \kappa \le \pi/2$, and

$$\mathbf{\Lambda} = \left(\begin{array}{ll} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right)$$

Computing the Jacobian of this transformation gives $|J| = \frac{1}{2}\left(\lambda_1-\lambda_2\right)^2\sin\left(2\kappa\right)$ and integrating over $d\mathbf{S}$ gives the formula (15) for the unordered joint PDF $p\left(\lambda_1,\lambda_2\right)$.

The same analysis can be done for the $t \times r$ case, but the number of integrals turns out to be very large and therefore the mathematical complexity becomes prohibitive. Note that for the Rayleigh case ($\sigma_X = \sigma_Y$), the results given in (13) and (15) reduce in a exact manner to the results already presented in [2, Eq. 94] and [2, Eq. 95], respectively. In the same way, for $b = 1$ ($\sigma_Y = 0$) the expressions (13) and (15) reduce in a exact manner to [2, Eq. 55] and [2, Eq. 58], respectively.

VI. The Asymptotic Case

For the asymptotic case, the result given in [9] can be used even when dealing with complex matrices with unequal real and imaginary parts. The PDF for the eigenvalue λ of the $\frac{1}{n}\mathbf{W}$ matrix is given by

$$p\left(\lambda\right) = \frac{1}{2\pi\lambda\beta\Omega}\sqrt{\left(b-\lambda\right)\left(\lambda-a\right)} \qquad a \le \lambda \le b \tag{16}$$

where $a = \Omega\left(1-\sqrt{\beta}\right)^2$, $b = \Omega\left(1+\sqrt{\beta}\right)^2$, $\beta = r/t$, and $\Omega = E\left[\left|H_{ij}-E[H_{ij}]\right|^2\right] = \sigma_X^2 + \sigma_Y^2$. Using (16) in (7), the following result can be obtained for the asymptotic Hoyt channel capacity

$$\frac{C}{m} \underset{r,t\to\infty}{\longrightarrow} \int_a^b \log_2\left(1+P\lambda\right)p\left(\lambda\right)\, d\lambda \tag{17}$$

Although this formula is asymptotic, it is shown by simulation that it is quite accurate, even for a small number of antennas.

VII. Numerical Results

In Fig. 1, we validate with Monte Carlo simulations the expression given in (15) for the joint eigenvalue distribution. In Fig. 1, we have plotted the marginal PDF for different values of the Hoyt parameter b: $b = 0$, $b = 0.5$ and $b = 0.9$. Note the excellent agreement between the simulation and the theoretical results.

In Fig. 2, the channel capacity is simulated and compared with the theoretical result (7) for the 2×2 case and $b = 0$,

$$p\left(\mathbf{W}\right)=\frac{K}{8}\,e^{-\frac{(w_1+w_2)}{4}\left(\frac{1}{\sigma_X^2}+\frac{1}{\sigma_Y^2}\right)}\int\limits_{\mathcal{Q}}e^{\left(\frac{\left(w_1^2+w_3^2-w_4^2\right)f_1(\theta,\phi_1,\phi_2)-2w_3w_4f_2(\theta,\phi_1,\phi_2)}{w_1}\right)\left(\frac{1}{4\sigma_Y^2}-\frac{1}{4\sigma_X^2}\right)}$$

$$\times\, e^{\left(\frac{\left(w_1w_2-w_3^2-w_4^2\right)f_3(\theta,\phi_1,\phi_2,\phi_3)+2\sqrt{w_1w_2-w_3^2-w_4^2}(w_4\cos(\phi_3)+w_3\sin(\phi_3))f_4(\theta,\phi_1,\phi_2)}{w_1}\right)\left(\frac{1}{4\sigma_Y^2}-\frac{1}{4\sigma_X^2}\right)}\sin\left(2\theta\right)d\mathbf{Q} \quad (13)$$

$$p\left(\lambda_1,\lambda_2\right)=\frac{K\left(\lambda_1-\lambda_2\right)^2}{32}\,e^{-\frac{(\lambda_1+\lambda_2)}{4}\left(\frac{1}{\sigma_X^2}+\frac{1}{\sigma_Y^2}\right)}\int\limits_{\mathcal{Q}}\int\limits_{\mathcal{S}}e^{\left(\frac{1}{4\sigma_Y^2}-\frac{1}{4\sigma_X^2}\right)f_1(\theta,\phi_1,\phi_2)\left(\lambda_1\cos(\kappa)^2+\lambda_2\sin(\kappa)^2\right)}\sin\left(2\theta\right)\sin\left(2\kappa\right)$$

$$\times\, e^{\left(\frac{4\lambda_1\lambda_2f_3(\theta,\phi_1,\phi_2,\phi_3)+(\lambda_1-\lambda_2)^2\sin(2\kappa)^2f_3(\theta,\phi_1,\phi_2,-\psi)+4(\lambda_1-\lambda_2)\sqrt{\lambda_1\lambda_2}f_4(\theta,\phi_1,\phi_2)\sin(2\kappa)\sin(\phi_3+\psi)}{4\left(\lambda_1\cos(\kappa)^2+\lambda_2\sin(\kappa)^2\right)}\right)\left(\frac{1}{4\sigma_Y^2}-\frac{1}{4\sigma_X^2}\right)}d\mathbf{S}d\mathbf{Q} \quad (15)$$

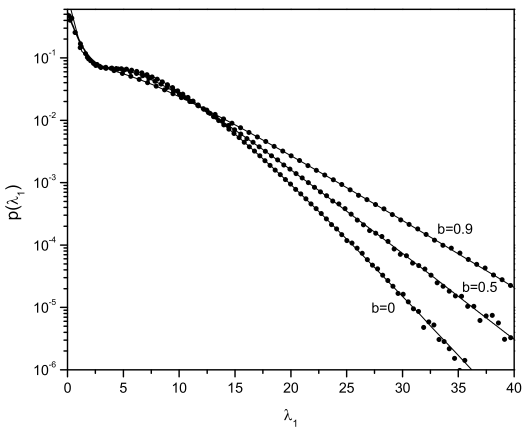

Fig. 1. Hoyt eigenvalue PDF for $b = 0,\ 0.5,\ 0.9$ ($\Omega = 2$).

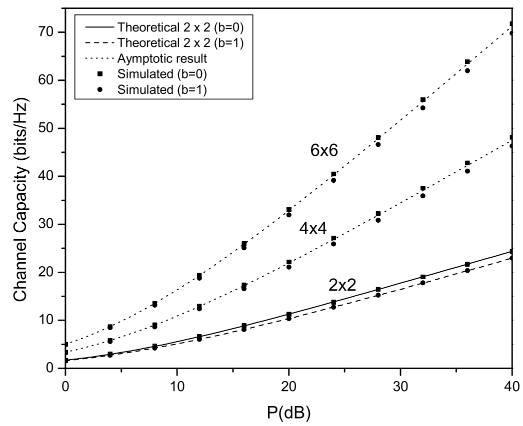

Fig. 2. The Hoyt channel capacity for limiting cases $b = 0, 1 (\Omega = 1)$.

$b = 1$. For the 4×4 and 6×6 case, simulations were performed and compared with the asymptotic result given in (16). The best performance is achieved for the Rayleigh case ($b = 0$) and reaches its minimum for $b = 1$ (one-sided Gaussian case). The effect of the Hoyt parameter b depends on the power P, but in the worst case it can degrade 8.5% of the capacity for the 2×2 case, 5% for the 4×4 case, and 3.5% for the 6×6 case.

Fig. 3, for P=5,10, and 15 dB, shows the theoretical and simulated channel capacity for the 2×2 case varying the Hoyt parameter b.

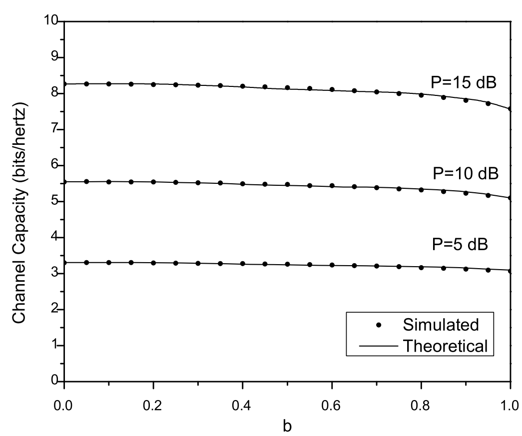

Fig. 3. The 2×2 Hoyt channel capacity as function of the parameter b ($\Omega = 1$).

VIII. Conclusion

In this paper, a new distribution, the Hoyt-Wishart distribution and its eigenvalue distribution were obtained. This distribution was used to compute the Hoyt 2×2 channel capacity and for this case the Hoyt parameter b can degrade 8.5% of the channel capacity. As the number of antennas increases, the effect of b is less perceptive. The asymptotic channel capacity was also addressed for the $t \times r$ case, and in all the cases, the results were validated by simulations.

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