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A probabilistic approach of sum rules for heat polynomials

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Abstract

In this paper, we show that the sum rules for generalized Hermite polynomials derived by Daboul and Mizrahi (2005 *J. Phys. A: Math. Gen.* **38** 427–48) and by Graczyk and Nowak (2004 *C. R. Acad. Sci., Ser. 1* **338** 849) can be interpreted and easily recovered using a probabilistic moment representation of these polynomials. The covariance property of the raising operator of the harmonic oscillator, which is at the origin of the identities proved in Daboul and Mizrahi and the dimension reduction effect expressed in the main result of Graczyk and Nowak are both interpreted in terms of the rotational invariance of the Gaussian distributions. As an application of these results, we uncover a probabilistic moment interpretation of two classical integrals of the Wigner function that involve the associated Laguerre polynomials.

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1. Introduction

1.1. Context and objectives

In a recent publication [1], Daboul *et al* derived several new sum rules for Hermite and generalized Hermite polynomials using an operational approach; most of these identities are a consequence of the invariance property of the raising operator of the harmonic oscillator under the group of complex rotations

$$O_{\mathbb{C}}(d) = \{M \in \mathbb{C}^{d\times d}; MM^t = M^tM = I_d\}.$$

In another publication, Graczyk *et al* [5] proposed a new multivariate sum rule for these generalized polynomials[1]; their proof is based on the analytical properties of the generating functions of these polynomials.

Our aim here is to show that these sum rules can be described as consequences of probabilistic properties of Gaussian vectors; we will show that this approach, together with

[1] The authors call them Gould–Hopper polynomials, but they should rather be called heat polynomials.

some basic techniques of probability calculus, allows elementary derivations and extensions of all these rules. As an application, we will use this probabilistic approach for the computation of two classical integrals of the Wigner function; this allows us to uncover some interesting relationships between these integrals, the noncentral chi-squared distribution and the associated Laguerre polynomials. Before addressing these identities, we introduce the main probabilistic tools and notations that will be needed for our purposes.

1.2. Moment representation of generalized Hermite polynomials

We will use the well-known integral representation of the Hermite polynomials [2, 8.951]

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (x + \imath z)^n \, \mathrm{e}^{-z^2} \, \mathrm{d}z$$

that we will express under its equivalent expectation form

$$H_n(x) = 2^n \mathbb{E}_z (x + \imath z)^n, \tag{1}$$

where z is Gaussian with zero mean and variance $\sigma^2 = \frac{1}{2}$, denoted as $z \sim \mathcal{N}\left(0, \frac{1}{2}\right)$. The expectation operator with respect to the distribution of a random variable z is denoted as $\mathbb{E}_z$.

Daboul *et al* consider the family of generalized Hermite polynomials

$$H_n(\alpha, \beta; x) = \left(\frac{\alpha\beta}{2}\right)^{\frac{n}{2}} H_n\left(\sqrt{\frac{\alpha}{2\beta}} x\right), \tag{2}$$

with α, β and $x \in \mathbb{C}$. Their moment representation is thus, with $z \sim \mathcal{N}\left(0, \frac{1}{2}\right)$,

$$H_n(\alpha, \beta; x) = \alpha^n \mathbb{E}_z \left(x + \imath\sqrt{\frac{2\beta}{\alpha}} z\right)^n. \tag{3}$$

This family includes, as special cases, several types of polynomials that are relevant in physics:

- the monomials

$$(\alpha x)^n = H_n(\alpha, 0, x);$$

- the heat polynomials [9]

$$v_n(x, t) = H_n(1, -2t; x);$$

- the physicists' Hermite polynomials

$$H_n(x) = H_n(2, 1; x) \text{ and}$$

- the probabilists' Hermite polynomials

$$\mathcal{H}_n(x) = H_n(1, 1; x).$$

We note that the moment representation (3) allows us to easily recover the three scaling properties derived in [1, equations (A5)–(A7)]: $\forall \sigma \neq 0$,

$$H_n(\alpha, \beta; x) = H_n(\sigma^{-1}\alpha, \sigma\beta; \sigma x) = \sigma^{-n} H_n(\sigma\alpha, \sigma\beta; x) = \sigma^{-n} H_n(\alpha, \sigma^2\beta; \sigma x).$$

Moreover, using (3) their generating function can be easily computed—without the use of the Baker–Campbell–Hausdorff formula as in [1]—as follows:

$$\sum_{n=0}^{+\infty} \frac{u^n}{n!} H_n(\alpha, \beta; x) = \mathbb{E}_z \sum_{n=0}^{+\infty} \frac{(2u)^n}{n!} \left(\frac{\alpha}{2} x + \imath\sqrt{\frac{\alpha\beta}{2}} z\right)^n = \mathbb{E}_z \exp(\alpha u x + \imath\sqrt{2\alpha\beta} u z).$$

Recognizing here the Gaussian characteristic function

$$\mathbb{E}_z \exp(\imath\sqrt{2\alpha\beta}uz) = \exp\left(-\frac{\alpha\beta}{2}u^2\right),$$

we deduce the generating function

$$\exp\left(\alpha ux - \frac{\alpha\beta}{2}u^2\right).$$

Most of the sum rules studied in the following will be considered for Hermite polynomials ($\alpha = 2, \beta = 1$), but they can be immediately extended to the generalized Hermite polynomials using the identity (2); for the sake of completion, their general case will be provided in appendix C.

1.3. Complex rotational invariance principle for the Hermite polynomials

The fundamental property needed in these proofs—the counterpart of the invariance property of the raising operators—is the invariance of the moments of the Gaussian distributions under the group of complex rotations. We thus need the following slight extension of (1).

Property 1. *Complex rotational invariance principle for the Hermite polynomials.*
The Hermite polynomial of degree n satisfies

$$H_n(x) = 2^n\mathbb{E}_{\mathbf{z}}(x + \imath(M\mathbf{z})_j)^n, \forall j \in \{1, \dots, d\}$$

for any $M \in O_{\mathbb{C}}(d)$ and any real Gaussian vector $\mathbf{z} \sim \mathcal{N}\left(0, \frac{1}{2}I_d\right)$.

The proof is given in appendix A. We also need the following property.

1.4. Rotational invariance principle for the Gaussian vectors

Property 2. *If $\mathbf{z}_1$ and $\mathbf{z}_2$ are independent Gaussian $\mathcal{N}(0, \sigma^2 I_d)$, then*

$$\mathbf{u}_1 = \frac{\mathbf{z}_1 + \mathbf{z}_2}{\sqrt{2}}, \qquad \mathbf{u}_2 = \frac{\mathbf{z}_1 - \mathbf{z}_2}{\sqrt{2}}$$

are again independent Gaussian $\mathcal{N}(0, \sigma^2 I_d)$.

This property can be checked, for example, by inspection of the characteristic functions of the vectors. The last property that we will need is the following.

1.5. Cancelation rule

Property 3. *If z_1 and z_2 are independent scalar Gaussian $\mathcal{N}(0, \sigma^2)$ and if f is an entire function, then*

$$\mathbb{E}_{z_1,z_2} f(x + z_1 + \imath z_2) = f(x), \qquad \forall x \in \mathbb{C}.$$

This property is a consequence of the fact that the monomial functions z^m form an orthogonal basis of the Fock–Bargmann Hilbert space equipped with the scalar product [10]

$$\langle f, g\rangle = \int_{\mathbb{C}} \bar{f}(z)g(z)\exp(-|z|^2)\,\mathrm{d}z\,\mathrm{d}\bar{z}.$$

Finally, we introduce the following notations.

1.6. Notations

Bold-typed variables denote vectors and capitalized variables denote matrices. The Pochhammer symbol is $(n)_j = \frac{\Gamma(n+j)}{\Gamma(n)}$, $|\mathbf{z}|$ is the Euclidean norm of the real-valued vector $\mathbf{z}$ while $\mathbf{z}^t$ is its transpose and $\sim$ denotes equality in distribution. We will also adopt the multi-index notation: with $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$,

$$\mathbf{z}^{\mathbf{n}} = \prod_{i=1}^{d} z_i^{n_i}, \mathbf{n}! = \prod_{i=1}^{d} n_i!$$

2. Probabilistic moments approach to sum rules

2.1. The factorization sum rule

The factorization sum rule for Hermite polynomials [1, equation (5)] is

$$2^n H_n\left(\frac{x+y}{\sqrt{2}}\right) H_n\left(\frac{x-y}{\sqrt{2}}\right) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k H_{2n-2k}(x) H_{2k}(y). \tag{4}$$

We note that this identity can be found in [3, 4.5.2.5].

It is proved in [1] using the covariant transformation of the raising operators for the harmonic oscillator under the group of complex rotations $O_{\mathbb{C}}(d)$. In the probabilistic framework, this property is replaced by the rotational invariance principle for Gaussian vectors as follows: the left-hand side of (4) can be written using (1), with u_1 and u_2 being independent $\sim \mathcal{N}\left(0, \frac{1}{2}\right)$, as

$$2^{3n} \mathbb{E}_{u_1}\left(\frac{x+y}{\sqrt{2}} + \imath u_1\right)^n \mathbb{E}_{u_2}\left(\frac{x-y}{\sqrt{2}} + \imath u_2\right)^n$$
$$= 2^{3n} \mathbb{E}_{z_1,z_2}\left(\frac{x+y}{\sqrt{2}} + \imath\frac{z_1+z_2}{\sqrt{2}}\right)^n \left(\frac{x-y}{\sqrt{2}} + \imath\frac{z_1-z_2}{\sqrt{2}}\right)^n,$$

where z_1 and z_2 are again independent $\sim \mathcal{N}\left(0, \frac{1}{2}\right)$, so that we obtain

$$2^{2n} \mathbb{E}_{z_1,z_2}((x+\imath z_1)^2 - (y+\imath z_2)^2)^n = 2^{2n} \mathbb{E}_{z_1,z_2} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (x+\imath z_1)^{2n-2k} (y+\imath z_2)^{2k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k H_{2n-2k}(x) H_{2k}(y),$$

which concludes the proof.

2.2. The generalized factorization sum rule

An extension of the previous sum rule reads [1, proposition 3], with $c, s \in \mathbb{C}$ and $c^2 + s^2 = 1$:

$$H_m(cx - sy) H_n(sx + cy) = \sum_{r=0}^{m+n} \alpha_r^{(m,n)}(c, s) H_r(x) H_{m+n-r}(y), \tag{5}$$

with coefficients $\alpha_r^{(m,n)}(c, s)$ given by

$$\alpha_r^{(m,n)}(c, s) = \sum_{k=0}^{n \wedge r} \binom{m}{r-k} \binom{n}{k} (-1)^{m-r+k} c^{n+r-2k} s^{m-r+2k}. \tag{6}$$

This result is again a direct consequence of the rotational invariance principle for Gaussian vectors; with the same notations as above, the left-hand side of (5) is

$$
\begin{aligned}
&2^{m+n}\mathbb{E}_{z_1,z_2}\left(cx-sy+\imath\left(cz_1-sz_2\right)\right)^m\mathbb{E}_{z_1,z_2}\left(sx+cy+\imath\left(sz_1+cz_2\right)\right)^n\\
&\quad=2^{m+n}\mathbb{E}_{z_1,z_2}\sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{n}{k}c^{l+n-k}\left(-1\right)^{m-l}s^{m+k-l}\\
&\qquad\times\left(x+\imath z_1\right)^{l+k}\left(y+\imath z_2\right)^{m-l+n-k}\\
&\quad=\sum_{l=0}^{m}\sum_{k=0}^{n}\binom{m}{l}\binom{n}{k}c^{l+n-k}\left(-1\right)^{m-l}s^{m+k-l}H_{k+l}(x)H_{m+n-k-l}(y)\\
&\quad=\sum_{r=0}^{m+n}\alpha_r^{(m,n)}\left(c,s\right)H_r(x)H_{m+n-r}(y),
\end{aligned}
$$

with coefficients $\alpha_r^{(m,n)}\left(c,s\right)$ given as in (6), which coincides with [1, equation (19)].

We were able to identify a fraction of these coefficients as follows.

Theorem 2.1. *For $r \leqslant m$ and $r \leqslant n$, the coefficients $\alpha_r^{(m,n)}\left(c,s\right)$ can be expressed as*

$$
\alpha_r^{(m,n)}\left(c,s\right)=\left(-1\right)^m c^{n-r}s^{m-r}P_r^{(n-r,m-r)}\left(s^2-c^2\right),
$$

where $P_r^{(m,n)}$ is the Jacobi polynomial of degree r with parameters m and n.

Proof. The Jacobi polynomial of degree r with parameters $m-r$ and $n-r$ reads

$$
P_r^{(m-r,n-r)}(x)=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}\left(\frac{x-1}{2}\right)^{r-k}\left(\frac{x+1}{2}\right)^{k}.
$$

The result is obtained by identifying $x=s^2-c^2$. □

2.3. The summation theorem

We show now that the summation theorem [1, proposition 1] can be viewed again as a consequence of the rotational invariance of Gaussian vectors. We give here an equivalent but a more compact version of this formula as follows: if $O\in O_{\mathbb{C}}(d)$, then

$$
\frac{1}{n!}H_n\left((O\mathbf{x})_i\right)=\sum_{|\mathbf{p}|=n}\frac{\mathbf{o}_i^{\mathbf{p}}}{\mathbf{p}!}H_{\mathbf{p}}\left(\mathbf{x}\right),
$$

where $\mathbf{o}_i$ is the ith column vector of O^t.

This result is easily proved, remarking that

$$
H_n\left((O\mathbf{x})_i\right)=H_n\left(\sum_{j=1}^{d}O_{ij}x_j\right)=2^n\mathbb{E}_{u_i}\left(\sum_{j=1}^{d}O_{ij}x_j+\imath u_i\right)^n=2^n\mathbb{E}_{\mathbf{z}}\left(\sum_{j=1}^{d}O_{ij}\left(x_j+\imath z_j\right)\right)^n,
$$

with $u_i=\sum_{j=1}^{d}O_{ij}z_j$. Using the multinomial expansion of the nth power, we obtain

$$
2^n\mathbb{E}_{\mathbf{z}}\sum_{|\mathbf{p}|=n}\frac{n!}{\mathbf{p}!}\prod_{j=1}^{d}O_{i,j}^{p_j}(x_j+\imath z_j)^{p_j}=\sum_{|\mathbf{p}|=n}\frac{n!}{\mathbf{p}!}\prod_{j=1}^{d}O_{i,j}^{p_j}H_{p_j}(x_j)
$$

which is the desired result.

2.4. Another sum rule based on rotational invariance

The sum rule [1, proposition 5] reads, with $\mathbf{w} = O\mathbf{x}$,

$$\sum_{|\mathbf{q}|=n} \frac{1}{\mathbf{q}!} H_{2\mathbf{q}}(\mathbf{w}) = \sum_{|\mathbf{p}|=n} \frac{1}{\mathbf{p}!} H_{2\mathbf{p}}(\mathbf{x}) . \tag{7}$$

The moment representation (1) allows us to identify the left-hand side, up to an $n!$ factor, as the multinomial expansion

$$\mathbb{E}_{\mathbf{z}} \sum_{|\mathbf{q}|=n} \frac{n!}{\mathbf{q}!} \prod_{j=1}^{d} 2^{2q_j} (w_j + \imath z_j)^{2q_j} = \mathbb{E}_{\mathbf{z}} \left(\sum_{j=1}^{d} \left(2w_j + 2\imath z_j\right)^2 \right)^n$$
$$= \mathbb{E}_{\mathbf{z}} (|2O\mathbf{x} + 2\imath\mathbf{z}|^2)^n.$$

By the rotational invariance principle, this is equal to

$$\mathbb{E}_{\mathbf{z}} (|2O\mathbf{x} + 2\imath O\mathbf{z}|^2)^n$$

which coincides, up to the same $n!$ factor, with the right-hand side.

2.5. A sum rule due to translation symmetry

This last sum rule

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} (2y)^{n-k} H_k(x) \tag{8}$$

is obtained in [1, equation (25), note that index s should start from 0] as a consequence of the translational invariance of the raising operator of the harmonic oscillator. We note also that this sum can be found in [3, 4.5.1.7].

We propose here an elementary proof as a consequence of the moment representation (1) and using the binomial formula as follows:

$$H_n(x+y) = 2^n \mathbb{E}_z (x+y+\imath z)^n = 2^n \mathbb{E}_z \sum_{k=0}^{n} \binom{n}{k} (x+\imath z)^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (2y)^{n-k} H_k(x).$$

This proof shows that the counterpart of the translation invariance of the raising operator of the harmonic oscillator is precisely the property that the Hermite polynomial can be represented under the form (1), namely

$$H_n(x) = 2^n \mathbb{E}_z (x + \imath z)^n.$$

This property itself is a consequence of the fact that the family of Hermite polynomials is a Scheffer-type family with a basic sequence $\{x^n\}$; for more details on this property, we refer the reader to [8].

3. An elaborated sum rule due to Graczyk and Nowak

3.1. An extended version

In [5], Graczyk and Nowak prove an interesting sum rule for the multivariate heat polynomials that does not seem to be related to the previous family of rules, and to the rotational invariance property of the Gaussian vectors. We show here that this rotational invariance is indeed at the basis of this result.

Let us first state Graczyk's identity for the multivariate heat polynomials, defined as

$$g_{\mathbf{m}}(\mathbf{x}, p) = \prod_{i=1}^{m} g_{m_i}(x_i, p).$$

The version for the generalized Hermite polynomials is given in appendix C.

Theorem 3.1. *For all* $m \in \mathbb{N}$ *and for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\sum_{|\mathbf{m}|=m} \frac{1}{\mathbf{m}!} g_{\mathbf{m}}(\mathbf{x}, p) g_{\mathbf{m}}(\mathbf{y}, p) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(2p)^{2j}}{j!\,(m-2j)!} \left(\frac{d-1}{2}\right)_j g_{m-2j}(x, p) g_{m-2j}(y, p), \tag{9}$$

with

$$x = \frac{|\mathbf{x}+\mathbf{y}| + |\mathbf{x}-\mathbf{y}|}{2}, \qquad y = \frac{|\mathbf{x}+\mathbf{y}| - |\mathbf{x}-\mathbf{y}|}{2}. \tag{10}$$

This result is proved by Graczyk and Nowak using properties of generating functions related to the heat polynomials. We provide here a probabilistic proof that allows us to interpret the two striking features of this identity, namely

– the dimensionality reduction phenomenon expressed by (10)
– the presence of the enigmatic Pochhammer term on the right-hand side of (9).

We need the following probabilistic tools.

3.2. Some new probabilistic tools

Our first tool is a stochastic representation of the inner product of two noncentered Gaussian vectors.

Lemma 3.2. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $p \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v}$ *are independent random vectors* $\sim \mathcal{N}(0, I_d)$*, then*

$$(\mathbf{x} + \sqrt{p}\mathbf{u})^t(\mathbf{y} + \sqrt{p}\mathbf{v}) \sim (x + \sqrt{p}u)(y + \sqrt{p}v) + p z_{d-1} w, \tag{11}$$

where u, v *and* w *are independent standard Gaussian random variables,*

$$x = \frac{|\mathbf{x}+\mathbf{y}| + |\mathbf{x}-\mathbf{y}|}{2}, \qquad y = \frac{|\mathbf{x}+\mathbf{y}| - |\mathbf{x}-\mathbf{y}|}{2} \tag{12}$$

and z_{d-1} *is chi-distributed with* $d-1$ *degrees of freedom (denoted as* $z_{d-1} \sim \chi_{d-1}$*) and independent of* u, v *and* w.

Proof. The proof is based on the polarization identity that allows us to express this inner product as

$$\begin{aligned}(\mathbf{x} + \sqrt{p}\mathbf{u})^t(\mathbf{y} + \sqrt{p}\mathbf{v}) &= \tfrac{1}{4}[|\mathbf{x} + \sqrt{p}\mathbf{u} + \mathbf{y} + \sqrt{p}\mathbf{v}|^2 - |\mathbf{x} + \sqrt{p}\mathbf{u} - \mathbf{y} - \sqrt{p}\mathbf{v}|^2] \\ &\sim \tfrac{1}{4}[|\mathbf{x} + \mathbf{y} + \sqrt{2p}\mathbf{u}_1|^2 - |\mathbf{x} - \mathbf{y} + \sqrt{2p}\mathbf{v}_1|^2],\end{aligned}$$

where vectors $\mathbf{u}_1 = \frac{1}{\sqrt{2}}(\mathbf{u}+\mathbf{v})$ and $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(\mathbf{u}-\mathbf{v})$ are again Gaussian and independent as a consequence of the rotational invariance principle.

The next step is again a consequence of the rotational invariance principle for Gaussian vectors; if $\mathbf{g} \sim \mathcal{N}(0, I_d)$ and $\mathbf{z} \in \mathbb{R}^d$ is a constant vector, then the norm of the random vector $\mathbf{z} + \mathbf{g}$ depends on $|\mathbf{z}|$ only; more precisely

$$|\mathbf{z} + \mathbf{g}| \sim ||\mathbf{z}|\mathbf{e}_1 + \mathbf{g}| = \sqrt{(|\mathbf{z}| + g_1)^2 + g_2^2 + \cdots + g_d^2},$$

where $\mathbf{e_1}$ is the first (or any) column vector of the identity matrix I_d. We deduce that

$$(\mathbf{x} + \sqrt{p}\mathbf{u})^t(\mathbf{y} + \sqrt{p}\mathbf{v}) \sim \tfrac{1}{4}[||\mathbf{x}+\mathbf{y}|\mathbf{e}_1 + \sqrt{p}\mathbf{u} + \sqrt{p}\mathbf{v}|^2 - ||\mathbf{x}-\mathbf{y}|\mathbf{e}_1 + \sqrt{p}\mathbf{u} - \sqrt{p}\mathbf{v}|^2]$$

and by the polarization identity, with x and y as in (12), this expression simplifies to

$$(x\mathbf{e}_1 + \sqrt{p}\mathbf{u})^t(y\mathbf{e}_1 + \sqrt{p}\mathbf{v}) = (x + \sqrt{p}u_1)(y + \sqrt{p}v_1) + p\sum_{i=2}^{d} u_i v_i.$$

The proof follows then from the identity in distribution, with $z_{d-1} \sim \chi_{d-1}$ independent of $w \sim \mathcal{N}(0, 1)$:

$$\sum_{i=2}^{d} u_i v_i \sim \left(\sum_{i=2}^{d} u_i^2\right)^{\frac{1}{2}} w = z_{d-1}w.$$

We now are in a position to prove Graczyk's theorem 3.1; using the moment representation of the generalized Hermite polynomials (3), we recognize the left-hand side of (9) as the multinomial expansion

$$\mathbb{E}_{\mathbf{u},\mathbf{v}} \sum_{|\mathbf{m}|=m} \frac{1}{\mathbf{m}!} \prod_{i=1}^{d} (x_i + \sqrt{2p}u_i)^{m_i}(y_i + \sqrt{2p}v_i)^{m_i}$$
$$= \frac{1}{m!}\mathbb{E}_{\mathbf{u},\mathbf{v}} \left(\sum_{i=1}^{d} (x_i + \sqrt{2p}u_i)(y_i + \sqrt{2p}v_i)\right)^m,$$

where $\mathbf{u}$ and $\mathbf{v} \sim \mathcal{N}(0, I_d)$ are independent. The sum is identified as the inner product

$$(\mathbf{x} + \sqrt{2p}\mathbf{u})^t(\mathbf{y} + \sqrt{2p}\mathbf{v}).$$

Now using the multinomial expansion of the mth power of the stochastic equivalent (11) of this expression and taking expectations, with $\mathbb{E}z_{d-1}^{2j} = 2^j \left(\frac{d-1}{2}\right)_j$ and $\mathbb{E}w^{2k} = \frac{(2k)!}{2^k k!}$ and $\mathbb{E}w^{2k+1} = 0$ for $w \sim \mathcal{N}(0, 1)$, we obtain the desired result. □

4. Application to the evaluation of Wigner functions

4.1. Laguerre polynomials and noncentral chi-squared distributions

In [1], the sum rules derived above are used to evaluate the two famous integrals of Wigner functions

$$I_n(x, p) = \int_{-\infty}^{+\infty} H_n(x + y)H_n(x - y)\exp(-y^2)\exp(2\imath py)\,\mathrm{d}y \tag{13}$$

and

$$I_{m,n}(x, y) = \int_{-\infty}^{+\infty} H_m(x + z)H_n(y + z)\exp(-z^2)\,\mathrm{d}z. \tag{14}$$

Both can be evaluated in terms of the *associated Laguerre* polynomials $L_n^{(m)}(x)$ and of the Laguerre polynomials $L_n(x) = L_n^{(0)}(x)$ as [2, 7.377 for the second integral]

$$I_n(x, p) = (-1)^n\sqrt{\pi}2^n n! L_n(2x^2 + 2p^2)$$

and

$$I_{m,n}(x, p) = \sqrt{\pi}2^n m! y^{n-m} L_m^{(n-m)}(-2xy),$$

respectively. Before we can propose a probabilistic approach to both evaluations, we need a preliminary characterization of those associated Laguerre polynomials. We recall the well-known

Definition 4.1. *A random variable x follows the noncentral chi-squared distribution with m degrees of freedom and noncentrality parameter* λ *(denoted as* $x \sim \chi^2_{m,\lambda}$*) if*

$$x = \sum_{i=1}^{m} (\mu_i + n_i)^2 ,$$

where $n_i \sim \mathcal{N}\left(0, \frac{1}{2}\right)$ *are independent*[2] *and* $\lambda = \sum_{i=1}^{m} \mu_i^2$.

By the rotational invariance principle, this distribution depends on λ *only so that*

$$x \sim (\sqrt{\lambda} + n_1)^2 + \sum_{i=2}^{m} n_i^2.$$

The Laguerre polynomials are related to the noncentral chi-squared distribution as follows (see for example [4]).

Proposition 4.2. *If* $x \sim \chi^2_{2m+2,\lambda}$, *then*

$$Ex^n = n! L_n^{(m)}(-\lambda), \;\; \lambda > 0, \;\; n \geqslant 0. \tag{15}$$

An immediate application of this moment representation is the Scheffer identity for Laguerre polynomials [7, page 110]

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_{n-k}^{(\alpha)}(x) L_k^{(\beta)}(y) \tag{16}$$

as a consequence of the stability property of noncentral chi-squared distributions: if $u \sim \chi^2_{2\alpha+2,x}$ and $v \sim \chi^2_{2\beta+2,y}$ are independent, then, again by the rotational invariance principle, $u + v \sim \chi^2_{2\alpha+2\beta+4,x+y}$. The polynomial identity (16) holding for all negative values of x and y holds thus for all complex valued x and y.

Another consequence of the moment representation (15) is the evaluation of the following sum rule, used by Schleich *et al* [6] in the study of the harmonic oscillator

$$\sum_{k=0}^{n} \binom{n}{k} H_{2n-2k}(x) H_{2k}(y) = (-2)^n n! L_n(2x^2 + 2y^2). \tag{17}$$

Using the moment representation (1), the left-hand side evaluated at $\imath x$ and $\imath y$ reads

$$2^{2n} \mathbb{E}_{z_1,z_2}((\imath x + \imath z_1)^2 + (\imath y + \imath z_2)^2)^n = (-4)^n \mathbb{E} u^n,$$

where $u \sim \chi_{2,x^2+y^2}$, which, by applying (15), yields the desired result.

We note that this proof uses basically the same ingredients as the factorization sum rule (4) so that, at least in this probabilistic framework, the similarity between both sum rules (4) and (17) is far from 'deceptive', as claimed by Daboul *et al*.

For the evaluation of integral (14), we will also need the famous derivation rule for associated Laguerre polynomials

$$\frac{\mathrm{d}}{\mathrm{d}x} L_n^{(m)}(x) = -L_{n-1}^{(m+1)}(x). \tag{18}$$

Surprisingly, and contrarily to the Hermite case where

$$\frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = n H_{n-1}(x)$$

is a direct consequence of the moment representation (1), the proof of (18) from the moment representation (15) seems difficult. In appendix B, we explain the reason of this difficulty, showing that the derivation rule (18) is the consequence of a deeper identity for the noncentral chi-squared distribution, namely a diffusion equation.

[2] We depart here from the usual definition by using a Gaussian with variance 1/2 instead of unit variance; this will make the link with Hermite polynomials easier to use.

4.2. Evaluation of the first Wigner integral

The computation of the Wigner integral

$$I_n(x,p) = \int_{-\infty}^{+\infty} H_n(x+y)H_n(x-y)\exp(-y^2)\exp(2\imath py)\,\mathrm{d}y,\ x,p\in\mathbb{R}$$

is performed in [1] using essentially the sum rule (8) and the Fourier transform of the wavefunction of the quantum harmonic oscillator.

We provide here a simpler probabilistic approach, remarking first that the integral $I_n\,(x,p)$ is an analytic function in each variable x or p so that we can compute first $I_n\,(\imath x, \imath p)$. Remarking next that

$$\exp(-y^2)\exp(-2py) = \exp(p^2)\exp(-(y+p)^2),$$

we deduce

$$\begin{aligned} I_n(\imath x,\imath p) &= \sqrt{\pi}2^{2n}\exp(p^2)\mathbb{E}_{z_1,z_2,y}[(\imath x+y-p+\imath z_1)^n(\imath x-y+p+\imath z_2)^n] \\ &= \sqrt{\pi}\exp(p^2)\mathbb{E}_{z_1,z_2,y}[(2\imath x+\imath z_1+\imath z_2)^2-(2y+2p+\imath z_1-\imath z_2)^2]^n, \end{aligned}$$

where we have applied the polarization identity. By the rotational invariance of the Gaussian, and expanding $2y\sim\sqrt{2}y_1+\sqrt{2}y_2$, where again y_1 and y_2 are Gaussian and independent, we deduce

$$I_n(\imath x,\imath p) = \sqrt{\pi}\exp(p^2)\mathbb{E}_{u_1,u_2,y_1,y_2}[-(2x+\sqrt{2}u_1)^2-(2p+\sqrt{2}y_1+\sqrt{2}y_2+\imath\sqrt{2}u_2)^2]^n;$$

applying the cancelation rule to the second term, we end up with

$$\sqrt{\pi}2^n(-1)^n\exp(p^2)\mathbb{E}_{u_1,y_1}[(\sqrt{2}x+u_1)^2+(\sqrt{2}p+y_1)^2]^n$$

and recognize the nth moment of a noncentral chi-squared random variable with the noncentrality parameter $2x^2+2p^2$; replacing x and p by $-\imath x$ and $-\imath p$ respectively yields the result.

4.3. Evaluation of the second Wigner integral

The second Wigner integral $I_{m,n}(x,y)$ considered in [1] is evaluated assuming $m\leqslant n$ without loss of generality. We start from

$$\mathbb{E}(x+z+\imath u)^m(y+z+\imath v)^n = \frac{m!}{n!}\frac{\partial^{n-m}}{\partial x^{n-m}}\mathbb{E}(x+z+\imath u)^n(y+z+\imath v)^n. \quad (19)$$

This moment can be computed using the polarization identity as

$$\begin{aligned} \mathbb{E}(x+z+\imath u)^n(y+z+\imath v)^n &= \frac{1}{2^{2n}}\mathbb{E}[(x+y+\sqrt{2}z_1+\sqrt{2}z_2+\imath\sqrt{2}u)^2-(x-y+\imath\sqrt{2}v)^2]^n \\ &= \frac{1}{2^n}\mathbb{E}\left[\left(\frac{x+y}{\sqrt{2}}+z_1\right)^2+\left(\frac{x-y}{\imath\sqrt{2}}+v\right)^2\right]^n \end{aligned}$$

after the application of the cancelation rule and rescaling. We recognize here the mth-order moment of a noncentral chi-squared distribution with the noncentrality parameter

$$\lambda = \left(\frac{x+y}{\sqrt{2}}\right)^2+\left(\frac{x-y}{\imath\sqrt{2}}\right)^2 = 2xy.$$

Hence,

$$\mathbb{E}\,(x+z+\imath u)^n\,(y+z+\imath v)^n = \frac{n!}{2^n}L_n\,(-2xy)\,.$$

Taking successive derivatives of this identity according to (19) and applying the derivation rule (18), we end up with

$$I_{m,n}(x,y) = \sqrt{\pi}2^{m+n}\frac{m!}{n!}\frac{1}{2^n}\frac{\partial^{n-m}}{\partial x^{n-m}}L_n\,(-2xy) = \sqrt{\pi}2^n y^{n-m}m!L_m^{(n-m)}\,(-2xy)$$

which is the desired result.

5. Conclusion

We have shown that the sum rules given in [1] and [5] can be deduced from a general rotational invariance principle for Gaussian vectors. The moment representation approach that we used is certainly not the only one, and one may think of using other approaches such as the following operational characterization of the generalized Hermite polynomials:

$$H_n(\alpha, \beta; x) = \exp\left(-\frac{\beta}{2\alpha}\frac{d^2}{\mathrm{d}x^2}\right)(\alpha x)^n,$$

which can be proved by expanding the exponential term and comparing it to the expansion of the generalized Hermite polynomials obtained from (3) as

$$H_n(\alpha, \beta; x) = \alpha^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!k!}\left(-\frac{\beta}{2\alpha}\right)^k x^{n-2k}.$$

Now, the rotational invariance of the Gaussian vectors is replaced by the rotational invariance of the Laplacian operator; if $O \in \mathcal{O}_{\mathbb{C}}(d)$ and $\mathbf{w} = O\mathbf{x}$, then for any multivariate polynomial f,

$$\Delta_{\mathbf{w}} f(\mathbf{w}) = \Delta_{\mathbf{x}} f(\mathbf{x}),$$

which gives another proof of such identities as (7).

One may also mention the umbral approach (see for example [11] for a good review); the Hermite umbra M is defined as

$$H_n(x) = (2x + M)^n,$$

with the implicit rule that

$$M^{2k} = (-1)^k \frac{(2k)!}{k!},$$

and $M^{2k+1} = 0$. However, we are not aware of any multivariate extension of this approach.

Appendix A. Proof of the complex rotational invariance principle

We only need to check that $(M\mathbf{z})_j$ has the same moments as a real Gaussian with zero mean and variance $\sigma^2 = \frac{1}{2}$. Let us compute the characteristic function

$$\begin{aligned}\mathbb{E}_z \exp(\imath (M\mathbf{z})_j) &= \mathbb{E}_{\mathbf{z}} \exp\left(\imath t \sum_{k=1}^{d} M_{jk} z_k\right) = \prod_{k=1}^{d} \mathbb{E}_{z_k} \exp(\imath t M_{jk} z_k) \\ &= \prod_{k=1}^{d} \exp\left(-\frac{t^2}{2} M_{jk}^2\right) = \exp\left(-\frac{t^2}{2}\sum_{k=1}^{d} M_{jk}^2\right).\end{aligned}$$

By the (complex) orthogonality condition $MM^t = M^t M = I_d$, this sum is equal to 1 and we are left with $\exp\left(-\frac{t^2}{2}\right)$, so that the characteristic function of the complex-valued random variable $(M\mathbf{z})_j$ coincides with that of the real-valued Gaussian z_j. Since

$$t \mapsto \mathbb{E}\exp(\imath t z_j) = \mathbb{E}\exp(\imath t (M\mathbf{z})_j)$$

is analytic in a neighborhood of 0, we deduce that $\mathbb{E} z_j^n = \mathbb{E}(M\mathbf{z})_j^n$ for all integers n.

Appendix B. A diffusion equation for the noncentral chi-squared distribution

We prove here the fact that the derivation rule (18) is a consequence of the following diffusion equation for the noncentral chi-squared distribution.

Lemma B.1. *The noncentral chi-squared probability density with m degrees of freedom and noncentrality parameter* λ*, denoted as* $f_{m,\lambda}$*, satisfy the diffusion equation*

$$\frac{\partial}{\partial\lambda} f_{m,\lambda}(x) = -\frac{\partial}{\partial x} f_{m+2,\lambda}(x). \tag{B.1}$$

As a consequence, the Laguerre polynomials satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} L_n^{(m)}(\lambda) = -L_{n-1}^{(m+1)}(\lambda). \tag{B.2}$$

Proof. The proof is based on the fact that the noncentral chi-squared distribution is a Poisson mixture of central chi-squared distributions,

$$f_{m,\lambda}(x) = \sum_{i=0}^{+\infty} \mathrm{Po}_{\frac{\lambda}{2}}(i)\, f_{m+2i,0}(x),$$

where $\mathrm{Po}_{\frac{\lambda}{2}}$ denotes the Poisson distribution with parameter $\frac{\lambda}{2}$,

$$\mathrm{Po}_{\frac{\lambda}{2}}(i) = \mathrm{e}^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^i}{i!}, \qquad i \geqslant 0.$$

It can be easily checked that the Poisson distribution satisfies the diffusion equation

$$\frac{\partial}{\partial\lambda} \mathrm{Po}_{\frac{\lambda}{2}}(i) = -\frac{1}{2}\mathrm{Po}_{\frac{\lambda}{2}}(i) + \frac{1}{2}\mathrm{Po}_{\frac{\lambda}{2}}(i-1), \qquad i \geqslant 0$$

and that the central chi-squared distribution satisfies the diffusion equation

$$\frac{\partial}{\partial x} f_{m,0}(x) = -\frac{1}{2} f_{m,0}(x) + \frac{1}{2} f_{m-2,0}(x),$$

hence the diffusion equation. Now, the associated Laguerre polynomial is the moment

$$L_n^{(m)}(-\lambda) = \frac{1}{m!} E\left(\chi^2_{2m+2,\lambda}\right)^n = \frac{1}{m!}\int_0^{+\infty} x^2 f_{m,\lambda}(x)\,\mathrm{d}x.$$

Taking the derivative with respect to λ and applying the diffusion equation (B.1) yields the result after a simple algebra. □

Appendix C. Sum rules for generalized Hermite polynomials

The above sum rules are given here for the generalized Hermite polynomials. Note that the first four identities are obtained by simply replacing the Hermite polynomial by its generalized version.

- The factorization sum rule

$$2^n H_n\left(\alpha, \beta; \frac{x+y}{\sqrt{2}}\right) H_n\left(\alpha, \beta; \frac{x-y}{\sqrt{2}}\right) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k H_{2n-2k}(\alpha, \beta; x) H_{2k}(\alpha, \beta; x).$$

- The generalized factorization sum rule

$$H_m(\alpha, \beta; cx - sy)\, H_n(\alpha, \beta; sx + cy) = \sum_{r=0}^{m+n} \alpha_r^{(m,n)}(c, s)\, H_r(\alpha, \beta; x) H_{m+n-r}(\alpha, \beta; y).$$

- The summation theorem

$$\frac{1}{n!}H_n\left(\alpha,\beta;(O\mathbf{x})_i\right)=\sum_{|\mathbf{p}|=n}\frac{\mathbf{o}_i^{\mathbf{p}}}{\mathbf{p}!}H_{\mathbf{p}}\left(\alpha,\beta;\mathbf{x}\right).$$

- Another sum rule based on rotational invariance, with $\mathbf{w}=O\mathbf{x}$

$$\sum_{|\mathbf{q}|=n}\frac{1}{\mathbf{q}!}H_{2\mathbf{q}}\left(\alpha,\beta;\mathbf{w}\right)=\sum_{|\mathbf{p}|=n}\frac{1}{\mathbf{p}!}H_{2\mathbf{p}}\left(\alpha,\beta;\mathbf{x}\right).$$

- A sum rule due to translation symmetry

$$H_n\left(\alpha,\beta;x+y\right)=\sum_{k=0}^{n}\binom{n}{k}(\alpha y)^{n-k}H_k(\alpha,\beta;x).$$

- Graczyk and Nowak's sum rule, with $\mathbf{x}$ and $\mathbf{y}$ as in (10)

$$\sum_{|\mathbf{m}|=m}\frac{1}{\mathbf{m}!}H_{\mathbf{m}}\left(\alpha,\beta;\mathbf{x}\right)H_{\mathbf{m}}\left(\alpha,\beta;\mathbf{y}\right)$$
$$=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\frac{(\alpha\beta)^{2j}}{j!\,(m-2j)!}\left(\frac{d-1}{2}\right)_j H_{m-2j}(\alpha,\beta;x)H_{m-2j}\left(\alpha,\beta;y\right).$$

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