Partially Random Matrices in Line-of-Sight Wireless Networks

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***Abstract*—We analyze the performance of distributed MIMO transmissions in wireless networks, under the assumption of a line-of-sight environment. This leads us to the study of a new class of random matrices which depend on a linear number of random variables. We characterize the number of their significant singular values and give an upper bound on the size of their largest singular value. This translates into estimates on the number of degrees of freedom for the MIMO transmission.**

I. Introduction

The aim of the present paper is to study the efficiency of distributed multiple-input multiple-output (MIMO) transmissions in a wireless network with homogeneously distributed nodes, under the following classical line-of-sight propagation model between node k and node j in the network:

$$h_{jk} = \frac{e^{2\pi i r_{jk}/\lambda}}{r_{jk}}. \tag{1}$$

In the above equation, λ is the carrier wavelength and r_{jk} is the internode distance. More precisely, we are interested in assessing the performance of a distributed MIMO transmission between two far away clusters of n nodes in the network, by establishing how the number of spatial degrees of freedom is affected by the above line-of-sight assumption.

From a mathematical point of view, the above $n \times n$ matrix H is an interesting object, as it is halfway between a purely random matrix with i.i.d. entries and a fully deterministic matrix. Indeed, the internode distances r_{jk} are themselves random just because the node positions are. Hence, there are only order n independent sources of randomness in H, as opposed to the classical i.i.d. fading model classically considered in MIMO communications where the fading matrix contains order n^2 independent sources of randomness. This basic i.i.d. assumption was first questioned by [1] in the context of wireless networks. It has been shown (see [2]) to remain valid in the regime where the cluster areas are large enough compared to the distance between them. This has for consequence that full degrees of freedom are available for communication between the two clusters in this case. In the present paper, our aim is to consider the case where the i.i.d. assumption fails to hold and the number of spatial degrees of freedom available for communication is limited.

To this end, a precise characterization of the spectrum of the above matrix H is needed, but a direct analysis of the spectrum reveals itself difficult. It turns out that in the regime of interest for us, namely when the inter-cluster distance is large, the matrix H may be reasonably well approximated, up to a constant factor, by the $n \times n$ matrix G defined as

$$g_{jk} = e^{-2\pi i m y_j z_k} \tag{2}$$

where y_j, z_k are i.i.d. $\mathcal{U}([0,1])$ random variables and m is a parameter that grows with n as $m = n^\delta$, for some fixed $1/2 < \delta \leq 1$. The justification of this approximation is discussed later in the present paper and also in detail in our former paper on the subject [3].

The matrix G shares with H the same feature that it is generated by order n independent sources of randomness only, but its nice structure makes it more amenable to analysis. In this paper, we establish two main results regarding the spectrum of G. The first one concerns the number of significant singular values of G, a quantity directly linked to the spatial degrees of freedom of the corresponding MIMO channel. We show below that this number is, up to a logarithmic factor, of order m in the regime where $m = n^\delta$ and $1/2 < \delta \leq 1$.

Our second contribution concerns the spectral radius (or largest eigenvalue) of the matrix GG^*/n. We show below that, in the same regime as above, this spectral radius does not exceed n/m, up to a logarithmic factor. Using then the fact that

$$n = \mathrm{tr}(GG^*/n) = \sum_{j=1}^{n} \lambda_j,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of GG^*/n, we conclude, combining the above two results, that all the m significant eigenvalues of GG^*/n are roughly of the same order n/m (otherwise they could not sum up to n).

II. Spatial degrees of freedom

[1] Let us consider two square clusters of area A separated by a distance d, one containing n transmitters and the other containing n receivers uniformly distributed in their respective clusters, as illustrated on Fig. 1.

[1]The material of this section is essentially retaken from [3].

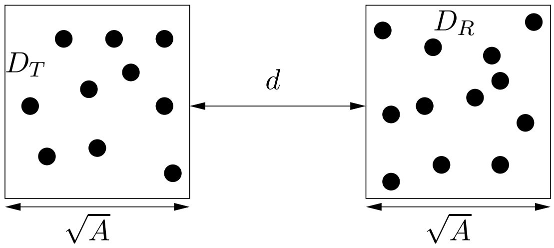

Fig. 1. Two square clusters of area A separated by distance d.

We are interested in estimating the number of spatial degrees of freedom of a MIMO transmission between the two clusters:

$$Y_j = \sum_k \sqrt{F}\, h_{jk}\, X_k + Z_j, \quad j = 1, \ldots, n,$$

where F is Friis' constant, the coefficients h_{jk} are given by the line-of-sight fading model (1) and Z_j represents additive white Gaussian noise at receiver j. The distance r_{jk} between node j at the receiver side and node k at the transmitter side is given by

$$r_{jk} = \sqrt{(d + \sqrt{A}\,(x_j + w_k))^2 + A\,(y_j - z_k)^2} \qquad (3)$$

where $x_j, w_k, y_j, z_k \in [0, 1]$ are normalized horizontal and vertical coordinates, as illustrated on Fig. 3.

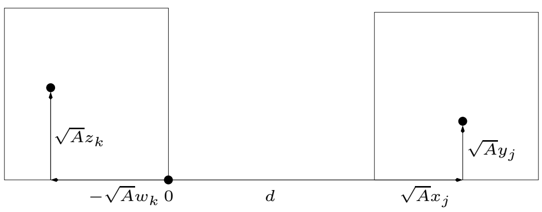

Fig. 2. Coordinate system.

Assuming full channel state information and perfect cooperation of the nodes on both sides, the maximum number of bits per second and per Hertz that can be transferred reliably from the transmit cluster to the receive cluster over this MIMO channel is given by the following expression:

$$C_n = \max_{Q \geq 0\,:\, Q_{kk} \leq P,\ \forall k} \log\det(I + HQH^*), \qquad (4)$$

where Q is the covariance matrix of the input signal vector $X = (X_1, \ldots, X_n)$ and P is the power constraint at each node (in order to simplify notation, we choose units so that the other parameters, such as Friis' constant F, the noise power spectral density N_0 and the bandwidth W do not appear explicitly in the above capacity expression).

In the sequel, we make the following *two assumptions*:

1) d, A both increase[2] with n and satisfy the relation $\sqrt{A} \leq d \leq A/\lambda$.

2) $P = \frac{(d+\sqrt{A})^2}{n}$; because the average distance between two nodes in opposite clusters is $d + \sqrt{A}$ and because the MIMO power gain is of order n, this power constraint ensures that the SNR of the incoming signal at each receiving node is of order 1 on average, so that the MIMO transmission operates at full power. Imposing this power constraint allows us to focus our attention on the spatial degrees of freedom of the system. Notice that because of assumption (1), the power P is allowed here to scale arbitrarily with n. This might sound a bit artificial, but as stated above, it serves our main goal, which is to assess the number of degrees of freedom of the system.

An upper bound on the capacity expression (4) can be found trivially by applying the Hadamard inequality to the determinant. This procedure leads to an upper bound of order n. A natural question is whether it is possible to tighten this bound on the capacity, hence showing that the distributed MIMO system does not have the full spatial degrees of freedom for its transmission. In order to answer this question, let us first observe that any matrix Q satisfying the above constraints also satisfies $Q \preceq nPI$ [3]. Thus,

$$C_n \leq \log\det(I + nPHH^*) = \sum_{k=1}^{n} \log(1 + \lambda_k) \qquad (5)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $nPHH^*$. The number of significant eigenvalues of $nPHH^*$ therefore determines the number of spatial degrees of freedom. The direct analysis of these eigenvalues appears to be difficult, so we proceed by approximating the matrix $nPHH^*$ by another matrix GG^*, easier to analyze.

Claim 1. *Let $m = A/(\lambda d)$ and G be the matrix whose entries are given by* (2)*:*

$$g_{jk} = e^{-2\pi i m y_j z_k},$$

where y_j, z_k, $1 \leq j, k \leq n$ are the same random variables as in expression (3)*. Then under assumptions 1) and 2), the following approximation holds:*

$$\log\det(I + nPHH^*) = \log\det(I + GG^*)\,(1 + o(1))$$

with high probability as n gets large.

Observe first that by expression (5), the above approximation is equivalent to saying that the number of significant eigenvalues of $nP\,HH^*$ and GG^* do not differ in order as n gets large. Some numerical evidence of this fact is provided on Fig. 4 for a given set of parameters (but a similar behavior is observed for a wide range of parameters).

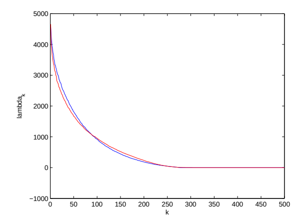

Fig. 3. Eigenvalues of $nPHH^*$ (blue) and GG^* (red) for the parameters $n = 500$, $A = 10'000m^2$, $d = 300m$, $\lambda = 0.1m$ (so $m = A/(\lambda d) \simeq 333$).

[2]By "increasing with n", we mean that $A = n^\beta$ and $d = n^\gamma$ for some powers $\beta, \gamma > 0$.

[3]where $A \preceq B$ denotes the partial ordering on the set of positive definite matrices

It can be observed on the figure that the eigenvalues drop to zero after a threshold of order $m = A/(\lambda d)$ for both matrices $nPHH^*$ and GG^*. The rest of the present section is devoted to the proof of this observation for the matrix G, which is recast in the following statement.

Theorem 2. *Let $m = n^\delta$, with $1/2 < \delta \leq 1$. Then there exists a constant $K_2 > 0$ such that*

$$\log\det(I + GG^*) \leq K_2\, m\, \log n$$

with high probability as n gets large.

Let us mention that applying the same technique as in [2], a matching lower bound on $\log\det(I + GG^*)$ can be found, up to logarithimic factors. This result on matrices G of the form (2) is interesting in itself, as these do not appear to have been studied before in the random matrix literature.

Provided Claim 1 holds true, this result also shows that the number of spatial degrees of freedom of a MIMO transmission between two clusters of area A separated by distance d is of order $m = A/(\lambda d)$, up to logarithmic factors (in the regime where $\sqrt{n} \ll m \leq n$).

Proof: An accompanying concentration result allows us to limit ourselves to computing the expected value. See [3] for a precise formulation.

In order to upperbound $\mathbb{E}(\log\det(I + GG^*))$, let us now expand the determinant, $G_{\mathcal{J}\times n}$ representing the matrix where only the rows in the index set $\mathcal{J}$ are present:

$$\begin{aligned}
&\mathbb{E}(\log\det(I + GG^*)) \\
&= \mathbb{E}\Bigg(\log\Bigg(1 + \sum_{k=1}^{n} \sum_{\substack{\mathcal{J}\subset\{1,\dots,n\}\\ |\mathcal{J}|=k}} \det(G_{\mathcal{J}\times n} G^*_{\mathcal{J}\times n})\Bigg)\Bigg) \\
&\leq \log\Bigg(1 + \sum_{k=1}^{n} \sum_{\substack{\mathcal{J}\subset\{1,\dots,n\}\\ |\mathcal{J}|=k}} \mathbb{E}(\det(G_{\mathcal{J}\times n} G^*_{\mathcal{J}\times n}))\Bigg)
\end{aligned}$$

where we used Jensen's inequality. Using the fact that the y_j are i.i.d., we further obtain that $\mathbb{E}(\det(G_{\mathcal{J}\times n} G^*_{\mathcal{J}\times n}))$ only depends on the size k of the subset $\mathcal{J}$, so

$$\begin{aligned}
&\mathbb{E}(\log\det(I + GG^*)) \\
&\leq \log\Bigg(1 + \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}(\det G_{k\times n} G^*_{k\times n})\Bigg) \\
&= \log\Bigg(1 + \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\Bigg(\sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=k}} \det(G_{k\times\mathcal{I}} G^*_{k\times\mathcal{I}})\Bigg)\Bigg) \\
&= \log\Bigg(1 + \sum_{k=1}^{n} \binom{n}{k}^2 \mathbb{E}(\det(G_{k\times k} G^*_{k\times k}))\Bigg)
\end{aligned}$$

where we have used this time the Cauchy-Binet formula together with the fact that the z_k are i.i.d. We thus see that in order to upperbound $\mathbb{E}(\log\det(I + GG^*))$, it is enough to control $\mathbb{E}(\det(G_{k\times k} G^*_{k\times k}))$, where $G_{k\times k}$ is the upper left $k \times k$ submatrix of G.

We will show that, similarly to what has been observed numerically for the eigenvalues λ_k, $\mathbb{E}(\det(G_{k\times k} G^*_{k\times k}))$ drops rapidly for k greater than a given threshold of order m, which will imply the result.

Using the definition of the determinant, we obtain

$$\begin{aligned}
&\mathbb{E}(\det(G_{k\times k} G^*_{k\times k})) \\
&= \sum_{\sigma,\tau\in S_k} (-1)^{|\sigma|+|\tau|}\, \mathbb{E}\Bigg(\prod_{j=1}^{k} g_{j,\sigma(j)}\, \overline{g_{j,\tau(j)}}\Bigg) \\
&= k! \sum_{\sigma\in S_k} (-1)^{|\sigma|}\, \mathbb{E}\Bigg(\prod_{j=1}^{k} g_{jj}\, \overline{g_{j,\sigma(j)}}\Bigg) \\
&= k! \sum_{\sigma\in S_k} (-1)^{|\sigma|}\, \mathbb{E}_Z\Bigg(\prod_{j=1}^{k} \mathbb{E}_Y(e^{-2\pi i y_j (z_j - z_{\sigma(j)})})\Bigg) \\
&= k! \sum_{\sigma\in S_k} (-1)^{|\sigma|}\, \mathbb{E}_Z\Bigg(\prod_{j=1}^{k} \frac{1 - e^{-2\pi i m(z_j - z_{\sigma(j)})}}{2\pi i m(z_j - z_{\sigma(j)})}\Bigg) \\
&= k!\, \mathbb{E}_Z\Bigg(\det\Bigg(\left\{ \frac{1 - e^{-2\pi i m(z_j - z_l)}}{2\pi i m(z_j - z_l)} \right\}_{1\leq j,l\leq k}\Bigg)\Bigg).
\end{aligned}$$

Multiplying row j by $e^{\pi i m z_j}$ and column l by $e^{-\pi i m z_l}$, we reduce the problem to computing the following determinant:

$$\mathbb{E}_Z\left(\det\left(\left\{ \frac{\sin(\pi m(z_j - z_l))}{\pi m(z_j - z_l)} \right\}_{1\leq j,l\leq k}\right)\right).$$

Operators and Fredholm Theory. The key observation is that the above expected value of the determinant can be seen as a classically studied quantity in the Fredholm theory of integral operators. This allows us to deduce precise estimates. A reference for the material discussed below is [4].

Consider the continuous function $K_m(x, y) = \frac{\sin(m(x-y))}{\pi(x-y)}$ on $[0, 1]^2$ and the associated operator $K_m : C([0, 1]) \to C([0, 1])$ defined as

$$K_m\, \phi(x) = \int_0^1 \frac{\sin(m(x-y))}{\pi(x-y)}\, \phi(y)\, dy.$$

The p^{th} iterated kernel K^p of an operator K is defined as $K^1 = K$ and

$$K^p(x, y) = \int_0^1 K^{p-1}(x, z)\, K(z, y)\, dz$$

Associated to this is the p^{th} trace of K:

$$A_p = \int_0^1 K^p(x, x)\, dx$$

Define as well the compound kernel $K_{[p]} \in C([0, 1]^{2p})$ as

$$K_{[p]}(\mathbf{x}, \mathbf{y}) = \det\begin{pmatrix} K(x_1, y_1) & K(x_1, y_2) & \dots & K(x_1, y_p) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_p, y_1) & K(x_p, y_2) & \dots & K(x_p, y_p)\end{pmatrix}$$

for $\mathbf{x} = (x_1, \dots, x_p)$ and $\mathbf{y} = (y_1, \dots, y_p)$. In this notation, the quantity we are interested in is

$$\mathbb{E}_Z\left(\det\left(\left\{ \frac{\sin(\pi m(z_j - z_l))}{\pi m(z_j - z_l)} \right\}_{1\leq j,l\leq k}\right)\right) = k!\, \frac{1}{m^k}\, d_k,$$

where

$$d_k = \frac{1}{k!} \int_{[0,1]^k} K_{m,[k]}(\mathbf{x},\mathbf{x})\, dx_1 \cdots dx_k.$$

Since K_m is a compact operator, it has a discrete spectrum $\mu_1 \geq \mu_2 \geq \ldots$ It is immediate to see that $A_p = \sum \mu_i^p$. Furthermore, the quantities d_k and A_p are related by the following recurrence relation, which follows from expanding the determinant in the definition of $K_{[p]}$ and regrouping equal terms (see [4]).

$$k\, d_k = \sum_{p=1}^{k} (-1)^{p-1} A_p\, d_{k-p}.$$

Using the fact that $A_p = \sum \mu_i^p$, it can be seen that the only solution to the above recurrence (with $d_0 = 1$) is

$$d_k = \sum_{i_1 < i_2 < \ldots < i_k} \mu_{i_1}\, \mu_{i_2} \cdots \mu_{i_k}. \tag{6}$$

We conclude that it is sufficient to estimate the eigenvalues of the operator K_m in order to upperbound d_k.

Since the kernel K_m is translation invariant, it can be defined equivalently on $[-1/2, 1/2]$, and this new operator has the same eigenvalues. This operator is called the sinc kernel and is well known in signal processing, since it is the Fourier transform of the indicator function. It was originally studied by D. Slepian in [5] and precise estimates exist on the behavior of its eigenvalues. In particular, we have the following recent result from [6, Theorem 3]:

Theorem 3. *Let $\delta > 0$. There exists $M \geq 1$ and $c > 0$ such that, for all $m \geq 0$ and $k \geq \max(M, cm)$,*

$$\mu_k \leq e^{-\delta\,(k-cm)}.$$

This theorem essentially says that the eigenvalues μ_k decay exponentially for $k \geq cm$. The direct consequence of this is that the numbers d_k decay as follows:

$$d_k \leq \begin{cases} 1, & \text{if } k \leq cm, \\ C^k\, e^{-\delta\,(k-cm)^2/2}, & \text{if } k > cm. \end{cases}$$

Gathering all estimates together, we finally obtain

$$\begin{aligned} &\mathbb{E}(\log\det(I + GG^*)) \\ &\leq \log\left(1 + \sum_{k=1}^{n} \binom{n}{k}^2 \frac{k!^2}{m^k}\, d_k\right) \leq \log\left(1 + \sum_{k=1}^{n} n^{2k}\, d_k\right) \\ &\leq \log\left(1 + \sum_{k=1}^{cm} n^{2k} + \sum_{k=cm+1}^{n} n^{2k}\, C^k\, e^{-\delta\,(k-cm)^2/2}\right) \\ &\leq (cm+1)\, \log n + O(1), \end{aligned}$$

which concludes the proof of Theorem 2. ■

III. Maximal Eigenvalue

Another quantity of interest is the maximal eigenvalue of the matrix GG^*/n. As already mentioned in the introduction, the following bound on the largest eigenvalue together with the results from the previous section allow us to conclude that all significant eigenvalues are roughly of the same order n/m in the regime of interest.

Theorem 4. *Let $m = n^\delta$ with $1/2 < \delta \leq 1$. Then there exists a constant $K_3 > 0$ such that*

$$\mathbb{E}(\lambda_{\max}(GG^*/n)) \leq K_3\, (n/m)\, \log n$$

and for every $\varepsilon > 0$,

$$\lambda_{\max}(GG^*/n) \leq K_3\, (n/m)\, n^\varepsilon$$

with high probability as $n \to \infty$.

Proof: We use the following inequality, valid for any integer $l \geq 1$:

$$\mathbb{E}(\lambda_{max}(GG^*/n)) \leq \mathbb{E}\left(\mathrm{t}r\left(\frac{GG^*}{n}\right)^l\right)^{1/l}, \tag{7}$$

in order to reduce the problem to computing

$$\frac{1}{n^l}\, \mathbb{E}\left(\sum_{\substack{1 \leq j_1, j_2, \ldots j_l \leq n \\ 1 \leq k_1, k_2, \ldots k_l \leq n}} g_{j_1 k_1} \overline{g_{j_2 k_1}} \ldots g_{j_l k_l} \overline{g_{j_1 k_l}}\right). \tag{8}$$

We will first concentrate on terms in this sum where all indices are different, i.e. $j_1 \neq j_2 \neq \ldots \neq j_l$ and $k_1 \neq k_2 \neq \ldots \neq k_l$. There are order n^{2l} such terms. Concretely, we get the following multiple integral for one term:

$$\frac{1}{n^l} \int_{[0,1]^{2l}} \{dy_j dz_k\}\, e^{-2\pi i m [z_1(y_2-y_1) + z_2(y_3-y_2) + \ldots + z_l(y_1-y_l)]} \tag{9}$$

For every j, we have

$$\int_0^1 dz_j\, e^{-2\pi i m z_j (y_{j+1}-y_j)} = \frac{1 - e^{-2\pi i m (y_{j+1}-y_j)}}{2\pi i m (y_{j+1}-y_j)}$$

so

$$\left|\int_0^1 dz_j\, e^{-2\pi i m z_j (y_{j+1}-y_j)}\right| \leq \max\left\{1, \frac{1}{\pi m} \frac{1}{|y_{j+1}-y_j|}\right\},$$

where the first term is obtained by simply noticing that the integrand on the left-hand side is upperbounded by 1. Observe next that

$$\int_0^1 dy_j\, 1_{\{|y_{j+1}-y_j| \geq \varepsilon\}}\, \frac{1}{\pi m} \frac{1}{|y_{j+1}-y_j|} \leq \frac{1}{\pi m} \log\left(\frac{1}{\varepsilon}\right).$$

The desired bound on (9) can then be obtained by first bounding the integral over one of the z_k term by 1 and then splitting the domain of integration to make sure the y_j's multiplying the same z_k are at least ε apart. One can then integrate over all remaining z_k's and repeat the above procedure $l-1$ times, as each integration reduces the occurrence of one of the y_j's. One then gets a bound of the form

$$\left(\frac{2\ln(1/\varepsilon)}{\pi m}\right)^{l-i-1}\varepsilon^i,$$

where i is the number of indices such that $|y_{j+1}-y_j|<\varepsilon$. Since there are $\binom{l}{i}$ ways to choose such indices, if one takes $\varepsilon=\frac{1}{m}$, one gets that (9) is bounded above by

$$\begin{aligned}
&\frac{1}{n^l}\sum_{i=0}^{l}\binom{l}{i}\left(\frac{2\ln(m)}{\pi m}\right)^{l-i-1}\left(\frac{1}{m}\right)^i\\
&\leq \frac{1}{n^l}\sum_{i=0}^{l}\binom{l}{i}\left(\frac{2\ln(m)}{\pi m}\right)^{l-1}\\
&\leq 2\,\frac{1}{n^l}\left(\frac{1}{m}\right)^{l-1}\left(\frac{4\ln(m)}{\pi}\right)^{l-1}.
\end{aligned}$$

As there are of the order of n^{2l} such terms, this gives us the desired bound after taking the l^{th} root.

We now take care of the case where some of the variables coincide. A minute's thought shows that we can reduce to the case where all the z variables are distinct. Observe that if i pairs of y variables are identified, there will be of the order of n^{2l-i} terms in the sum of (8), and so to get the same bound as in the previous section, we can afford to bound i of the z variables by 1 in the original integral (9). We now introduce a combinatorial way to look at the problem. Consider a graph whose vertices are the variables Y_j and with an edge between two vertices if the Y variables appear in front of the same Z variable in (9), after possible identifications of Y variables. For example, when all the Y_j's are different, we get a circle. We then delete one edge (the equivalent of bounding a term by 1 in the original integral as in the previous paragraph) and get a tree, which allows one to use successive integration. What remains to be seen is that given this circular graph, after identifying i pairs of vertices, one can get a tree by removing i edges. This is immediate.

Combining these results, we obtain

$$\begin{aligned}
&\mathbb{E}\left((\lambda_{max}(GG^*/n))^l\right)\leq \mathbb{E}\left(\mathrm{tr}\left(\frac{HH^*}{n}\right)^l\right)\\
&\leq \frac{1}{n^l}\,\mathbb{E}\left(\sum_{\substack{1\leq j_1,j_2,\dots j_l\leq n\\ 1\leq k_1,k_2,\dots k_l\leq n}} g_{j_1k_1}\overline{g_{j_2k_1}}\cdots g_{j_lk_l}\overline{g_{j_1k_l}}\right)\\
&\leq \frac{1}{n^l}\sum_{i=0}^{l}\binom{l}{i}\,2\left(\frac{4\ln(m)}{\pi m}\right)^{l-i-1}n^{2l-i}\\
&\leq C\,2^{l+1}\left(\frac{4\ln(m)n}{\pi m}\right)^{l-1}n,
\end{aligned}$$

where the index i counts the number of y indices that are paired up. Taking the l^{th} root of the last expression and using Jensen's inequality allows us to obtain the desired upper bound on the expectation in the theorem.

The second inequality may be obtained using Markov's inequaility:

$$\begin{aligned}
\mathbb{P}\left(\lambda_{max}(GG^*/n)\geq K_3(n/m)\,n^{\varepsilon}\right) &\leq \frac{\mathbb{E}\left((\lambda_{max}(GG^*/n))^l\right)}{(K_3(n/m))^l\,n^{\varepsilon l}}\\
&\leq \frac{(\log n)^l}{n^{\varepsilon l}},
\end{aligned}$$

which, for any fixed $\varepsilon>0$, can be made arbitrarily small by taking l sufficiently large. ■

IV. Conclusion and perspectives

The goal of this work is to give precise estimates on the number of spatial degrees of freedom in large MIMO systems in a line-of-sight environment. An upper bound for a model closely related to the line-of-sight model has been given, and the similarity of the two models is supported numerically. As such, it remains to be shown that the eigenvalues of the two models are indeed very close, in order to bound $|\log\det(I+nPHH^*)-\log\det(I+GG^*)|$; this is work in progress. Besides, numerical simulations suggest that more precise estimates on both the number of significant eigenvalues of GG^*/n and its maximal eigenvalue could probably be obtained. Finally, let us mention that the matrix G studied in this paper is related to Vandermonde matrices and random DFT matrices that appear in other contexts in the literature on wireless communications [7], [8], [9], [10] and compressed sensing [11], [12], respectively.

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