On the MIMO Channel Capacity for the Nakagami-m Channel

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***Abstract*— This paper presents the MIMO channel capacity over the Nakagami-m fading channel. The joint eigenvalue density function of $\mathbf{W} = \mathbf{H}\mathbf{H}^\dagger$, where H is the channel matrix, is derived in a closed form in the 2×2 case and for integer values of m, as well as for $m \to \infty$. The marginal eigenvalue distribution of W is also derived in closed form solution. For the more general $r \times t$ case, an asymptotic formulation is presented and is shown to be close to simulations, even for a small number of antennas. All the results are validated by numerical Monte Carlo simulations and are in excellent agreement.**

***Index Terms*— Fading distributions, Rayleigh distribution, Nakagami-m distribution, Eigenvalue distribution, MIMO channels.**

I. INTRODUCTION

It has been acknowledged in recent years that the use of multiple inputs and multiple outputs (MIMO) can potentially provide large spectral efficiency for wireless communications in the presence of multipath fading environments. In the pioneering paper presented by Telatar [1], the capacity was in particular shown to scale linearly with the number of antennas.

In most previous research on MIMO capacity, the channel fading is assumed to be Rayleigh distributed. Of course, the Rayleigh fading model is known to be a reasonable assumption for the fading encountered in many wireless communications systems. Nevertheless, many measurements campaigns [2], [3] show that the Nakagami-m distribution provides a much better fitting for the fading channel distribution. In fact, since the Nakagami-m distribution has one more free parameter, it allows for more flexibility. It moreover contains both the Rayleigh distribution ($m = 1$) and the uniform distribution on the unit circle ($m \to \infty$) as special (extreme) cases.

The Nakagami-m distribution is a general, but approximate solution to the random phase problem [4]. The exact solution to this problem involves the knowledge of the distribution and the correlations of all of the partial waves composing the total signal and becomes infeasible due to its complexity [5]. This has been circumvented by Nakagami [4] who, through empirical methods based on field measurements followed by a curve-fitting process, obtained the approximate distribution.

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Since the publication of [1], other distributions have been considered, as in [6] for the Ricean case, and recently [7] addressed the Hoyt distribution case.

The MIMO channel capacity can be computed by means of the joint eigenvalue density function (JEDF) of the matrix $\mathbf{W} = \mathbf{H}\mathbf{H}^\dagger$, where † denotes the complex conjugate transpose. In the classical Rayleigh fading model, the entries of $\mathbf{H}$ are assumed to be i.i.d. zero mean complex Gaussian, resulting in a matrix $\mathbf{W}$ which is Wishart distributed. This model takes advantages of the numerous results provided by the literature [8], [9]. Unfortunately, as one departs from this model, there is not too much that can be said.

This paper presents the MIMO channel capacity over the Nakagami-m fading channel. Channel state information is assumed at the receiver side only. In this case, it is shown that the uniform power distribution across the transmitting antennas maximizes the channel capacity. Assuming that the entries of $\mathbf{H}$ are i.i.d. with Nakagami-m distributed envelopes and uniform phases, an elegant and simple expression for the JEDF of the 2×2 matrix $\mathbf{W} = \mathbf{H}\mathbf{H}^\dagger$ is derived. The general $r \times t$ case is analyzed by means of the asymptotic formulation. All the results are validated by numerical Monte Carlo simulations and are shown to be in excellent agreement.

The paper is organized as follows: Sec. II introduces the Nakagami distribution in more detail. Sec. III defines channel capacity. Sec. IV presents the main result for the 2×2 case and Sec. V presents the asymptotic result. Finally, Sec. VI presents some numerical simulations and Sec. VII draws the conclusion.

II. THE NAKAGAMI-m DISTRIBUTION

The entries of the $r \times t$ channel matrix $\mathbf{H}$ are assumed to be i.i.d. and distributed as

$$Z = R\exp(j\Theta) \tag{1}$$

where the phase Θ is uniformly distributed and independent of the envelope R. R is in turn given by

$$R^2 = \sum_{i=1}^{m} X_i^2 + Y_i^2 \tag{2}$$

where X_i and Y_i are i.i.d. zero mean Gaussian distributed with variance $\Omega/2m$. The distribution of R is therefore the Nakagami-m distribution [4] given by

$$p_R(r) = \frac{2m^m r^{2m-1}}{\Omega^m \Gamma(m)} \exp\left(-\frac{mr^2}{\Omega}\right) \tag{3}$$

where $\Omega = \mathbb{E}\left[R^2\right]$, $m = \mathbb{E}[R^2]^2/\text{Var}[R^2]$, and $\Gamma(\cdot)$ denotes the Euler Gamma function. In addition, $p_{R,\Theta}(r,\theta) = p_R(r)\frac{1}{2\pi}$, yielding

$$p_{R,\Theta}(r,\theta) = \frac{m^m r^{2m-1}}{\pi\,\Omega^m \Gamma(m)} \exp\left(-\frac{mr^2}{\Omega}\right) \tag{4}$$

Note that $p_{R,\Theta}$ reduces to the Rayleigh distribution for $m=1$ and to the uniform distribution on the circle of radius $\sqrt{\Omega}$ for $m \to \infty$. The above family of distributions therefore allows to interpolate between the classical Rayleigh distribution and the "pure random phase" distribution.

Using the standard polar-rectangular transformation, the joint distribution of the real and imaginary parts of Z is given by $p_{X,Y}(x,y) = \frac{p_{R,\Theta}(r,\theta)}{r}$, thus

$$p_{X,Y}(x,y) = \frac{m^m \left(x^2+y^2\right)^{m-1}}{\pi\,\Omega^m \Gamma(m)}\, e^{-\frac{m\left(x^2+y^2\right)}{\Omega}} \tag{5}$$

and the marginals $p_X(x)$ and $p_Y(y)$ are given by

$$p_X(u) = p_Y(u) = \frac{e^{-\frac{mu^2}{\Omega}} m^m u^{2m} \Omega^{-m}}{\pi\Gamma(m)} \left(\frac{\Omega^{m-\frac{1}{2}}\Gamma\left(m-\frac{1}{2}\right)\, {}_1F_1\left(1-m;\frac{3}{2}-m;\frac{mu^2}{\Omega}\right) m^{\frac{1}{2}-m}}{u^{2m}} + \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}-m\right)\, {}_1F_1\left(\frac{1}{2};m+\frac{1}{2};\frac{mu^2}{\Omega}\right)}{u\Gamma(1-m)}\right) \tag{6}$$

where ${}_1F_1(\cdot)$ is the Kummer confluent hypergeometric function [10].

III. MIMO Channel Capacity

The following MIMO single-user Gaussian channel is considered, with t antennas at the transmitter and r antennas at the receiver:

$$\mathbf{y} = \mathbf{H}\,\mathbf{x} + \mathbf{n} \tag{7}$$

$\mathbf{H}$ is the $r \times t$ channel matrix with i.i.d. entries h_{ij}, each distributed as the random variable Z defined in (1).

The vector $\mathbf{y} \in \mathcal{C}^r$, $\mathbf{x} \in \mathcal{C}^t$, and $\mathbf{n}$ is zero-mean complex Gaussian noise with $\mathbb{E}\left[\mathbf{n}\,\mathbf{n}^\dagger\right] = \mathbf{I}$. In addition, a total transmit power constraint $\mathbb{E}[\mathbf{x}^\dagger \mathbf{x}] \le P$ is assumed.

For a given input covariance matrix $\mathbf{Q}$, the MIMO instantaneous capacity is given by

$$C(\mathbf{Q}) = \log_2 \det\left(\mathbf{I} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger\right) \tag{8}$$

and the MIMO channel capacity in the absence of channel knowledge at the transmitter is given by [1]

$$C = \sup_{\mathbf{Q}\ge 0:\text{tr}[\mathbf{Q}]\le P} \mathbb{E}\left[C(\mathbf{Q})\right] \tag{9}$$

Since the entries h_{ij} are i.i.d. and the distribution of h_{ij} is the same as that of $-h_{ij}$ for all i,j, one obtains from [11, Corollary 1b] that the uniform power allocation over the t transmit antennas maximizes the ergodic channel capacity. The capacity is therefore given by

$$C = \mathbb{E}\left[\log_2 \det\left(\mathbf{I} + \frac{P}{t}\mathbf{H}\,\mathbf{H}^\dagger\right)\right] \tag{10}$$

Defining then

$$\mathbf{W} = \begin{cases} \mathbf{H}\mathbf{H}^\dagger & r < t \\ \mathbf{H}^\dagger\mathbf{H} & r \ge t \end{cases} \tag{11}$$

and $n = \min\{r,t\}$, the capacity can be written in terms of the (unordered) eigenvalues $\lambda_1,\ldots,\lambda_n$ of $\mathbf{W}$ as

$$\begin{aligned} C &= \mathbb{E}\left[\sum_{i=1}^{n} \log_2\left(1+(P/t)\,\lambda_i\right)\right] && (12) \\ &= n\,\mathbb{E}\left[\log_2\left(1+(P/t)\,\lambda\right)\right] && (13) \end{aligned}$$

where λ is an eigenvalue of $\mathbf{W}$ picked uniformly at random among $\lambda_1,\ldots,\lambda_n$. Denoting by $p(\lambda)$ the distribution of λ, one obtains that

$$C = n\int_0^\infty \log_2\left(1+(P/t)\,\lambda\right)\, p(\lambda)\, d\lambda.$$

The purpose of the subsequent analysis is to provide an explicit expression for both the JEDF $p(\lambda_1,\ldots,\lambda_n)$ of the matrix $\mathbf{W}$ and the marginal distribution $p(\lambda)$. Note that $p(\lambda_1,\ldots,\lambda_n)$ and $p(\lambda)$ are related by

$$p(\lambda) = \int_{\mathbf{R}_+^{n-1}} p(\lambda,\lambda_2,\ldots,\lambda_n)\, d\lambda_2\cdots d\lambda_n. \tag{14}$$

IV. Main result for the 2×2 case

Theorem 1: The JEDF of the 2×2 matrix $\mathbf{W} = \mathbf{H}\mathbf{H}^\dagger$, where the entries of $\mathbf{H}$ are i.i.d. with Nakagami-m envelope and uniform phase, is given by

$$p(\lambda_1,\lambda_2) = K_{22}\, e^{-\frac{m(\lambda_1+\lambda_2)}{\Omega}}\, (\lambda_1-\lambda_2)^2\, F(\lambda_1,\lambda_2) \tag{15}$$

where $K_{22} = \left(\frac{m^m}{\pi\Omega^m\Gamma(m)}\right)^4$, $F(\lambda_1,\lambda_2)$ is given by (16) and $f(i_1,i_2,k_1,k_2)$ is given by (17).

In order to gain some intuition on the above result, note that the function $F(\lambda_1,\lambda_2)$ is a homogeneous polynomial of order $4(m-1)$. In the Rayleigh case ($m=1$), F is a constant and one recovers the classical JEDF of a Wishart matrix [8]:

$$p(\lambda_1,\lambda_2) = \frac{1}{2\Omega^4}\, e^{-\frac{(\lambda_1+\lambda_2)}{\Omega}}\, (\lambda_1-\lambda_2)^2 \tag{18}$$

The effect of the Nakagami-m envelope distribution on the JEDF is therefore expressed by the polynomial $F(\lambda_1,\lambda_2)$.

To illustrate the effect of the parameter m, Fig. 1 shows the resulting marginal eigenvalue distribution $p(\lambda)$ for $m = 1,2,10,\infty$ (see also Corollary 2). It can be observed that as the parameter m increases, the eigenvalues concentrate on the interval $[0,4\Omega]$, with higher probability on the boundary of the interval. It can be also seen on Fig. 1 that the result is in perfect agreement with Monte-Carlo simulations.

Proof: In order to get the JEDF of $\mathbf{W}$, the following steps need to be performed: 1) the joint distribution of $\mathbf{H}$ can be easily found, since its entries are i.i.d.; 2) the matrix $\mathbf{H}$ is decomposed as $\mathbf{H} = \mathbf{L}\mathbf{Q}$, where $\mathbf{L}$ is a complex lower triangular matrix with real positive diagonals and $\mathbf{Q}$ is a complex unitary matrix $\left(\mathbf{Q}\mathbf{Q}^\dagger = \mathbf{I}\right)$; 3) therefore, $\mathbf{W} = \mathbf{L}\mathbf{L}^\dagger$; 4) finally, performing the eigenvalue decomposition $\mathbf{W} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^\dagger$, one obtains the JEDF of $\mathbf{W}$.

$$F(\lambda_1,\lambda_2)=\frac{\pi^4}{\Gamma(4m-2)}\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-1}\sum_{k_1=0}^{i_1}\sum_{k_2=0}^{i_2} f(i_1,i_2,k_1,k_2)\,(\lambda_1\lambda_2)^{\frac{1}{2}(i_1+i_2-2(k_1+k_2))+m-1}\,(\lambda_1-\lambda_2)^{-i_1-i_2+2(k_1+k_2+m-1)} \tag{16}$$

$$f(i_1,i_2,k_1,k_2)=\frac{\left(\begin{array}{c} i_1\\ k_1\end{array}\right)\left(\begin{array}{c} i_2\\ k_2\end{array}\right)\left(\begin{array}{c} m-1\\ i_1\end{array}\right)\left(\begin{array}{c} m-1\\ i_2\end{array}\right)(-1)^{-i_1-2i_2+3m+1}\,2^{-2(k_1+k_2+1)}\Gamma\left(-\frac{i_1}{2}-\frac{i_2}{2}+m-\frac{1}{2}\right)}{\Gamma\left(-\frac{i_1}{2}-\frac{i_2}{2}+m\right)\Gamma\left(-\frac{i_1}{2}-\frac{i_2}{2}+k_1+k_2+m+\frac{1}{2}\right)}$$
$$\times\,\Gamma\left(-\frac{i_1}{2}-\frac{i_2}{2}+k_1+k_2+m\right)\Gamma\left(-\frac{i_1}{2}+\frac{i_2}{2}+k_1-k_2+2m-1\right)\Gamma\left(\frac{i_1}{2}-\frac{i_2}{2}-k_1+k_2+2m-1\right) \tag{17}$$

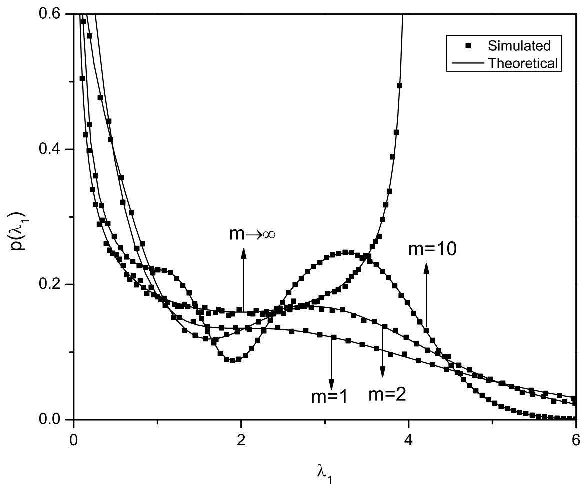

Fig. 1. The Nakagami eigenvalue distribution function for $m=1,2,10,100$ $(\Omega=1)$.

Since the entries of the channel matrix $\mathbf{H}$ are independent, their joint distribution is given by

$$p(\mathbf{H})=K_{22}\exp\left(-\frac{m\,\mathrm{tr}\left(\mathbf{H}\mathbf{H}^{\dagger}\right)}{\Omega}\right)\prod_{i,j=1}^{2}|h_{ij}|^{2(m-1)} \tag{19}$$

Using the LQ decomposition, the matrix $\mathbf{H}$ can be written as $\mathbf{H}=\mathbf{L}\mathbf{Q}$, where the matrix $\mathbf{Q}$ is given by [12]

$$\mathbf{Q}=\left(\begin{array}{cc} e^{j\phi_1}\cos(\theta) & e^{j\phi_2}\sin(\theta)\\ -e^{j(\phi_3-\phi_2)}\sin(\theta) & e^{j(\phi_3-\phi_1)}\cos(\theta)\end{array}\right) \tag{20}$$

and the variables are defined in the following range $0\leq\phi_1,\phi_2,\phi_3\leq 2\pi$, $0\leq\theta\leq\pi/2$. The matrix $\mathbf{L}$ is given by

$$\mathbf{L}=\left(\begin{array}{cc} l_{11} & 0\\ l_{21R}+j\,l_{21I} & l_{22}\end{array}\right) \tag{21}$$

The Jacobian of this transformation is given by [13] $J=l_{11}^3 l_{22}\sin(\theta)\cos(\theta)$, so the joint probability density function (PDF) of $\mathbf{L}$ and $\mathbf{Q}$ is given by

$$p(\mathbf{L},\mathbf{Q})=\frac{K_{22}}{2^{2m-1}}\,e^{\left(-\frac{m\,\mathrm{tr}\left(\mathbf{L}\,\mathbf{L}^{\dagger}\right)}{\Omega}\right)}\,l_{11}^{4m-1}l_{22}\sin^{2m-1}(2\theta)$$
$$\left(l_{22}^2\cos^2(\theta)+T_{1L}\sin^2(\theta)-T_{2L}\sin(2\theta)\right)^{m-1}$$
$$\left(T_{1L}\cos^2(\theta)+l_{22}^2\sin^2(\theta)+T_{2L}\sin(2\theta)\right)^{m-1} \tag{22}$$

where

$$T_{1L}=\left(l_{21I}^2+l_{21R}^2\right) \tag{23}$$

and

$$T_{2L}=l_{22}\left(l_{21I}\sin(\phi_1+\phi_2-\phi_3)-l_{21R}\cos(\phi_1+\phi_2-\phi_3)\right) \tag{24}$$

Since m is integer, it is possible to use the classical binomial expansion in (22). Therefore

$$p(\mathbf{L},\mathbf{Q})=\frac{K_{22}}{2^{2m-1}}\,e^{\left(-\frac{m\,\mathrm{tr}\left(\mathbf{L}\,\mathbf{L}^{\dagger}\right)}{\Omega}\right)}\,l_{11}^{4m-1}l_{22}$$
$$\times\sum_{i_1=0}^{m-1}\sum_{k_1=0}^{i_1}\sum_{i_2=0}^{m-1}\sum_{k_2=0}^{i_2}\left(\begin{array}{c} m-1\\ i_1\end{array}\right)\left(\begin{array}{c} i_1\\ k_1\end{array}\right)\left(\begin{array}{c} m-1\\ i_2\end{array}\right)\left(\begin{array}{c} i_2\\ k_2\end{array}\right)$$
$$\times(-1)^{m-1-i_2}T_{1L}^{k_1+k_2}T_{2L}^{2m-2-i_1-i_2}l_{22}^{2(i_2-k_2+i_1-k_1)}$$
$$\times\cos^{2(k_1+i_2-k_2)}(\theta)\,\sin^{2(i_1-k_1+k_2)}(\theta)\,\sin(2\theta)^{4m-3-i_1-i_2} \tag{25}$$

Integrating now over θ, ϕ_1, ϕ_2 and ϕ_3, the distribution of $p(\mathbf{L})$ is obtained in (26). The next transformation is given by

$$\mathbf{W}=\mathbf{L}\mathbf{L}^{\dagger}=\left(\begin{array}{cc} w_1 & w_3-jw_4\\ w_3+jw_4 & w_2\end{array}\right) \tag{27}$$

where the Jacobian of this change is given by $J=4l_{11}^3l_{22}$. Using this transformation, the distribution of $p(\mathbf{W})$ can be easily obtained.

The next step it to apply the eigenvalue decomposition $\mathbf{W}=\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{\dagger}$, where the matrix $\mathbf{S}$ is given by

$$\mathbf{S}=\left(\begin{array}{cc} \cos(\kappa) & -e^{i\psi}\sin(\kappa)\\ e^{-j\psi}\sin(\kappa) & \cos(\kappa)\end{array}\right) \tag{28}$$

$0\leq\psi\leq 2\pi$, $0\leq\kappa\leq\pi/2$, and $\mathbf{\Lambda}$ is the eigenvalue matrix given by

$$\mathbf{\Lambda}=\left(\begin{array}{cc} \lambda_1 & 0\\ 0 & \lambda_2\end{array}\right) \tag{29}$$

$$p(\mathbf{L}) = \frac{K_{22}\, 8\, \pi^{5/2}}{2^{2m-1}}\, e^{-\frac{m\,\mathrm{tr}(\mathbf{LL}^{\dagger})}{\Omega}} l_{11}^{4m-1} \sum_{i_1=0}^{m-1} \sum_{k_1=0}^{i_1} \sum_{i_2=0}^{m-1} \sum_{k_2=0}^{i_2} \binom{m-1}{i_1} \binom{i_1}{k_1} \binom{m-1}{i_2} \binom{i_2}{k_2}$$
$$\times (-1)^{m-1-i_2} \left(l_{21I}^2 + l_{21R}^2\right)^{k_1+k_2+\frac{1}{2}(2m-i_1-i_2-2)} l_{22}^{i_2-2k_2+i_1-2k_1+2m-1} \Gamma\left(\frac{1}{2}(2m-i_1-i_2-1)\right)$$
$$\times \frac{2^{4m-4-i_1-i_2}\Gamma\left(\frac{1}{2}(4m+2k_1+i_2-2k_2-2-i_1)\right)}{\Gamma(4(m-1))} \frac{\Gamma\left(\frac{1}{2}(i_1-2k_1+2k_2+4m-2-i_2)\right)}{\Gamma\left(\frac{1}{2}(2m-i_1-i_2)\right)} \tag{26}$$

The Jacobian of this last transformation can be obtained as $|J| = \frac{1}{2}(\lambda_1 - \lambda_2)^2 \sin(2\kappa)$. Finally, applying this transformation and integrating over ψ and κ, the JEDF (15) is obtained. ∎

The following corollary is a direct consequence of Theorem 1 and equation (14).

Corollary 2: The marginal distribution $p(\lambda)$ is given by

$$p(\lambda) = \frac{K_{22}\pi^4}{\Gamma(4m-2)} \times \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \sum_{i_3=0}^{2(k_1+k_2+m)-i_1-i_2} f_1(i_1,i_2,k_1,k_2) \times f_2(i_1,i_2,k_1,k_2,i_3)\, e^{-\frac{m\lambda}{\Omega}} \lambda^{\frac{1}{2}(i_1+i_2+2(m+i_3-k_1-k_2-1))} \tag{30}$$

where $f_2(i_1,i_2,k_1,k_2,i_3)$ is given by (31).

A. Special Cases: $m = 1, 2$ and ∞

As already mentioned, the JEDF (15) specializes to the classical Wishart distribution (18) in the case $m = 1$. For $m = 2$, the JEDF (15) specializes to

$$p(\lambda_1,\lambda_2) = \frac{4e^{-\frac{2(\lambda_1+\lambda_2)}{\Omega}}}{225\pi^3\Omega^8} (\lambda_1-\lambda_2)^2 \times \left(\lambda_1^4 + \lambda_2\lambda_1^3 + 26\lambda_2^2\lambda_1^2 + \lambda_2^3\lambda_1 + \lambda_2^4\right) \tag{32}$$

and the corresponding marginal eigenvalue distribution (30) is given by

$$p(\lambda) = \frac{4e^{-\frac{2\lambda}{\Omega}}}{225\Omega^7} \left(4\lambda^6 - 2\Omega\lambda^5 + 50\Omega^2\lambda^4 -150\Omega^3\lambda^3 + 150\Omega^4\lambda^2 - 15\Omega^5\lambda + 45\Omega^6\right) \tag{33}$$

The case $m \to \infty$ requires a slightly different analysis, which leads to the result below.

Theorem 3: The marginal eigenvalue distribution of the 2×2 matrix $\mathbf{W} = \mathbf{HH}^{\dagger}$, where $\mathbf{H}$ has i.i.d. entries with Nakagami-∞ envelope and uniform phase, is given by

$$p(\lambda) = \frac{1}{\pi\sqrt{4\Omega\lambda - \lambda^2}}\, 1_{0<\lambda<4\Omega} \tag{34}$$

Proof: As $m \to \infty$, the joint distribution of (R,Θ) given in (4) tends to the uniform distribution on the circle of radius $\sqrt{\Omega}$, therefore

$$\mathbf{H} = \sqrt{\Omega} \begin{pmatrix} e^{j\theta_{11}} & e^{j\theta_{12}} \\ e^{j\theta_{21}} & e^{j\theta_{22}} \end{pmatrix} \tag{35}$$

where θ_{ij} are uniformly distributed between 0 and 2π. In this case, the matrix $\mathbf{W}$ is given by

$$\mathbf{W} = \Omega \begin{pmatrix} 2 & w_{12} \\ \overline{w_{12}} & 2 \end{pmatrix} \tag{36}$$

where $w_{12} = e^{j(\theta_{11}-\theta_{21})} + e^{j(\theta_{12}-\theta_{22})}$ and $\overline{w_{12}}$ denotes its conjugate. Using the decomposition $\mathbf{W} = \mathbf{S\Lambda S}^{\dagger}$, it is possible to write $\lambda_1 + \lambda_2 = \mathrm{tr}(\mathbf{W}) = 4\Omega$ as well as $\lambda_1\lambda_2 = \Omega^2\left(4 - |w_{12}|^2\right)$, so

$$\lambda_1 = \Omega\left(2 \pm |w_{12}|\right) \tag{37}$$

and $\lambda_2 = 4\Omega - \lambda_1$. Note therefore that in this case, the correlation between the eigenvalues λ_1 and λ_2 is much stronger than in the finite m case. In particular, the joint eigenvalue density $p(\lambda_1,\lambda_2)$ does not exist.

The modulus of w_{12} can be written as $|w_{12}| = \sqrt{2\left(1+\cos(\theta_{11}-\theta_{21}-\theta_{12}+\theta_{22})\right)}$. Since the distribution of $\cos(\theta_{ij})$ is the same as the distribution of $y = \cos(\theta_{11} - \theta_{21} - \theta_{12} + \theta_{22})$, the distribution of this term is $p(y) = \frac{1}{\pi\sqrt{1-y^2}}\, 1_{|y|<1}$. Making the transformation of variable and computing the distribution of $|w_{12}|$, the following distribution is obtained

$$p\left(|w_{12}|\right) = \frac{1}{\pi\sqrt{1 - \frac{|w_{12}|^2}{4}}}\, 1_{|w_{12}|<1} \tag{38}$$

From (37) and (38), (34) follows directly. ∎

In Fig. 1, the expression (34) is compared with simulations for $m = 100$, and a nearly perfect match is observable.

V. Asymptotic Case

In the case where the number of antennas grows to infinity, the result given in [14] can be used, since the entries of $\mathbf{H}$ are i.i.d. The limiting eigenvalue distribution of the matrix $\frac{1}{n}\mathbf{W}$ is given by

$$p(\lambda) = \frac{1}{2\pi\lambda\beta\Omega}\sqrt{(b-\lambda)(\lambda-a)} \qquad a \le \lambda \le b \tag{39}$$

where $a = \Omega\left(1-\sqrt{\beta}\right)^2$, $b = \Omega\left(1+\sqrt{\beta}\right)^2$, $\beta = r/t$. Using (39) in (13), the following result is obtained for the asymptotic Nakagami channel capacity

$$\frac{C}{n} \underset{r,t\to\infty}{\to} \int_a^b \log_2(1+P\lambda)\, p(\lambda)\, d\lambda \tag{40}$$

Using the result presented in [15], one obtains

$$\frac{C}{n} \underset{r,t\to\infty}{\to} \frac{1}{\beta\ln 2}\left(\beta \ln\left(1+P-P\,v(\beta,P)\right) + \ln\left(1+P\beta - P\,v(\beta,P)\right) - v(\beta,P)\right) \tag{41}$$

where $v(\beta,P) = \frac{1}{2}\left(1+\beta+\frac{1}{P} - \sqrt{\left(1+\beta+P^{-1}\right)^2 - 4\beta}\right)$.

Although this formula is asymptotic, it is shown below by simulation that it is quite accurate, even for a small number of antennas.

$$f_2(i_1,i_2,k_1,k_2,i_3) = \begin{pmatrix} -i_1 - i_2 + 2\,(m+k_1+k_2) \\ i_3 \end{pmatrix} (-1)^{2m-i_1-i_2-i_3+2k_1+2k_2} \\ m^{\frac{1}{2}(-6m+i_1+i_2+2i_3-2k_1-2k_2)}\, \Omega^{3m-\frac{i_1}{2}-\frac{i_2}{2}-i_3+k_1+k_2}\, \Gamma\left(3m - \frac{i_1}{2} - \frac{i_2}{2} - i_3 + k_1 + k_2\right) \quad (31)$$

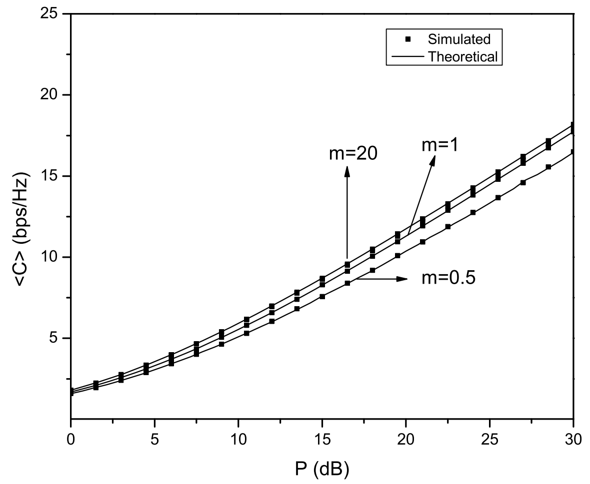

Fig. 2. The Nakagami channel capacity for the 2×2 case ($\Omega = 1$).

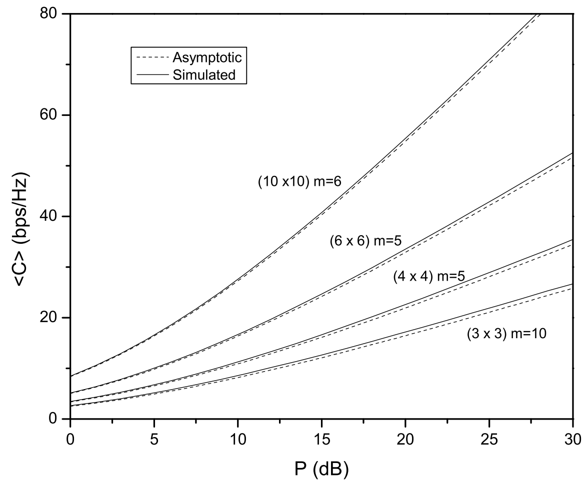

Fig. 3. The Asymptotic Nakagami channel capacity ($\Omega = 1$)

VI. Numerical Results

As already presented, Fig. 1 validates with Monte Carlo simulations the expression given in (15) for the JEDF for the cases $m = 1, 2, 10, \infty$. Note that there is an excellent agreement between the simulations and the theoretical results.

Fig. 2 compares the simulated channel capacity to the theoretical result (13) for the 2×2 case and $m = 0.5, 1, 20$. As can be seen on the figure, when m increases, the channel capacity also increases. The difference between the case $m = 0.5$ and $m = 20$ is small for low values of the power P, and increases as the power increases. The effect of the Nakagami parameter m depends on the power P, but in the worst case it can degrade 12.5% of the capacity for the 2×2 case. In Fig. 3, for the 3×3, 4×4, 6×6, and 10×10 cases for different values of m, simulations are performed and compared with the asymptotic result given in (41).

VII. Conclusion

In this paper, a general JEDF for the Nakagami-m channel is presented in a closed form solution for the 2×2 case. The marginal distribution is also derived in a closed form solution, and for the $m \to \infty$ case, a simple and elegant expression is obtained. For the more general $r \times t$ case, an asymptotic result is presented. The ergodic MIMO channel capacity is computed for the Nakagami-m channel, and for the 2×2 case, the effect of the parameter m is shown to degrade 12.5% the channel capacity. In all the cases, the results are validated by Monte Carlo simulations.

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