On the Broadcast Capacity of Large Wireless Networks at Low SNR

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***Abstract*—The present paper focuses on the problem of broadcasting information in the most efficient manner in a large two-dimensional *ad hoc* wireless network at low SNR and under line-of-sight propagation. A new communication scheme is proposed, where source nodes first broadcast their data to the entire network, despite the lack of sufficient available power. The signal's power is then reinforced via successive back-and-forth beamforming transmissions between different groups of nodes in the network, so that all nodes are able to decode the transmitted information at the end. This scheme is shown to achieve asymptotically the broadcast capacity of the network, which is expressed in terms of the largest singular value of the matrix of fading coefficients between the nodes in the network. A detailed mathematical analysis is then presented to evaluate the asymptotic behavior of this largest singular value.**

***Index Terms*—wireless networks, broadcast capacity, low SNR**

I. Introduction

The literature on the study of scaling laws in large *ad hoc* wireless networks concentrates mainly on *multiple-unicast* (one-to-one) transmissions (see e.g. [1], [2], [3]). This does not degrade by any means the importance of investigating *multicast* (one-to-many) transmissions for several reasons such as the need of many network protocols to broadcast control signals or to enhance cooperation among nodes belonging to the same cluster or cell. In the present paper, we are interested in studying how can source nodes broadcast their data to the whole network in the most efficient way. A few previous works investigate the broadcast capacity of wireless networks under specific channel models and mainly at high SNR [4], [5], [6]. Of course, multiple strategies exist in this context, but from the scaling law point of view (that is, for large networks), the simplest communication strategy, where source nodes take turns broadcasting their messages to the entire network, can be shown to be asymptotically optimal (up to logarithmic factors), when the power path loss is that of free space propagation. For a stronger power path loss, still at high SNR, simple multi-hopping strategies also allow to achieve an asymptotically optimal broadcast capacity, so there is not much to be discussed either in this case from the scaling law point of view.

In the present paper, we address the low SNR regime and consider the line-of-sight (LOS) propagation model described in Section II below. In this regime, the power available does not allow for a source node to successfully transmit a message to its nearest neighbor without waiting for some amount of time in order to spare power. In this case, contrary to the high SNR case, none of the two strategies described above (time-division or multi-hop broadcasting) is asymptotically optimal. This issue was first revealed in [7] in the context of one-dimensional networks, under the LOS model. For such networks, the authors proposed a hierarchical beamforming scheme to broadcast data to the network, that was proven to achieve asymptotic optimal performance. The generalization of this idea to two-dimensional networks is not immediate. Indeed, a particular feature of one-dimensional networks is that it is always possible for a group of nodes to beamform a given signal to all the other nodes in the network *simultaneously*. In two dimensions, a full beamforming gain is only achievable between groups of nodes that are sufficiently far apart from each other. This was already observed in [8], where a strategy was developed to enhance multiple-unicast communications in wireless networks under the LOS model. Taking inspiration from this paper, we propose below a new multi-stage beamforming scheme which is shown to achieve asymptotic optimal performance for broadcasting information in a two-dimensional wireless network. As it will be clear in the description of the scheme proposed, the nodes do not require any channel state information (CSI) except for their own location in the network. This fact adds an extra value to the practicality of the proposed scheme.

We give a detailed description of the scheme in Section III and a proof of its optimality in Section IV, that goes along with a general upper bound on the broadcast capacity of wireless networks at low SNR (see Theorem IV.1). This bound is shown to be proportional to the square of the largest singular value of the matrix made of fading coefficients between the nodes in the network. We then proceed to characterize the broadcast capacity of two-dimensional wireless networks. For this, we provide a detailed study of the largest singular value of the matrix under the LOS model. The study of this matrix reveals itself difficult, as there is much less randomness in such a matrix than in classical random matrices studied in the mathematical literature. In Section IV, we propose a recursive method to tackle this problem. We lack so far a complete rigorous proof, but our analysis is supported by both an approximation argument, as well as numerical simulations.

II. Model

There are n nodes uniformly and independently distributed in a square of area $A = n$, so that the node density remains constant as n increases. Every node wants to broadcast a different message to the whole network, and all nodes want to communicate at a common *per user* data rate r_n bits/s/Hz. We denote by $R_n = n\, r_n$ the resulting *aggregate* data rate and will often refer to it simply as "broadcast rate" in the sequel. The broadcast capacity of the network, denoted as C_n, is defined as the maximum achievable aggregate data rate R_n. We assume that communication takes place over a flat channel with bandwidth W and that the signal $Y_j[m]$ received by the j-th node at time m is given by

$$Y_j[m] = \sum_{k\in\mathcal{T}} h_{jk}\, X_k[m] + Z_j[m]$$

where $\mathcal{T}$ is the set of transmitting nodes, $X_k[m]$ is the signal sent at time m by node k and $Z_j[m]$ is additive white circularly symmetric Gaussian noise (AWGN) of power spectral density $N_0/2$ Watts/Hz. We also assume a common average power budget per node of P Watts, which implies that the signal X_k sent by node k is subject to an average power constraint $\mathbb{E}(|X_k|^2) \leq P$. In a line-of-sight environment, the complex baseband-equivalent channel gain h_{jk} between transmit node k and receive node j is given by

$$h_{jk} = \sqrt{G}\, \frac{\exp(2\pi i r_{jk}/\lambda)}{r_{jk}} \tag{1}$$

where G is Friis' constant, λ is the carrier wavelength, and r_{jk} is the distance between node k and node j. Let us finally define

$$\mathrm{SNR}_s = \frac{GP}{N_0 W}$$

which is the SNR available for a communication between two nodes at distance 1 in the network.

We focus in the following on the low SNR regime, by which we mean, as in [7], that $\mathrm{SNR}_s = n^{-\gamma}$ for some constant $\gamma > 0$. In order to simplify notation, we choose new measurement units such that $\lambda = 1$ and $G/(N_0 W) = 1$ in these units. This allows us to write in particular that $\mathrm{SNR}_s = P$.

III. Back-and-Forth Beamforming Strategy

First note that under the LOS model (1) and the assumptions made in the previous section, the time division scheme described in the introduction achieves a broadcast (aggregate) rate R_n of order $\min(P, 1)$. Indeed, a rate of order 1 is obviously achieved at high SNR[1]. At low SNR, each node can spare power while the others are transmitting, so as to compensate for the path loss of order $1/n$ between the source node and other nodes located at distance at most $\sqrt{2n}$, leading to a broadcast rate of order $R_n \sim \log(1 + n \times P/n) \sim \min(P, 1)$. As we will see, this broadcast rate is not optimal in the low SNR regime where $P = n^{-\gamma}$ with $\gamma > 0$.

In the following, we propose a new broadcasting scheme that will prove to be order-optimal. Our strategy is based on a back-and-forth beamforming technique between clusters. The scheme is split into two phases:

[1]We coarsely approximate $\log P$ by 1 here!

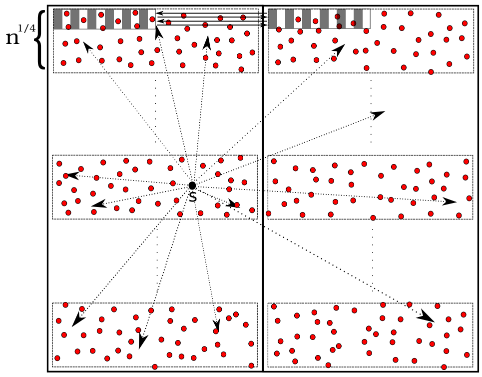

Fig. 1. An extended network divided into clusters of size $M = \frac{n^{1/4}}{2c} \times \frac{n^{1/2}}{4}$. As such, there are $2cn^{1/4}$ rows, each row containing 4 clusters. Source node S transmits its message to the whole network. Afterwards, clusters separated by distance $n^{1/2}/4$ pair up and start back-and-forth beamforming.

Phase 1. Broadcast Transmission. The source node broadcasts its message to the whole network. All the nodes receive a weak noisy signal in this phase. This phase requires only one time slot.

Phase 2. Back-and-Forth Beamforming with Time Division. Upon receiving the signal from the broadcasting node, clusters of size $\Theta(n^{1/4} \times n^{1/2})$ start back-and-forth beamforming transmissions, as illustrated on Fig. 1. During each transmission, there are $\Theta(n^{3/4})$ nodes participating on each side. This process is therefore repeated $\Theta(n^{1/4})$ times in a time division fashion to ensure the transmission of the message to the whole network. Assuming that the broadcast rate between corresponding clusters is $\Theta(n^{3/4-\epsilon}P)$ (to be shown below), we obtain that phase 2 requires $\Theta(n^{1/4} \times n^{-3/4+\epsilon}P^{-1})$ time slots. As such, back-and-forth beamforming achieves a broadcast rate of $\Theta(n^{1/2-\epsilon}P)$ bits per time slot.

In view of the described scheme, we are able to state the following result.

Theorem III.1. *For any $\epsilon > 0$ and $P = O(n^{-3/4})$, the following broadcast rate*

$$R_n = \Omega\left(n^{\frac{1}{2}-\epsilon}P\right)$$

is achievable with high probability in the network.

We first present the following two lemmas, the proofs of which can be found in [9], but they are also provided in the Appendix for completeness.

Lemma III.2. *Let us partition the network of area n into clusters of area M, with $M = n^{\beta}$ for some $0 < \beta < 1$. The number of nodes inside each cluster is between $((1-\delta)M,\ (1+\delta)M)$ with probability larger than $1 - \frac{n}{M}\exp(-\Delta(\delta)M)$ where $\Delta(\delta)$ is independent of n and satisfies $\Delta(\delta) > 0$ for $\delta > 0$.*

Lemma III.3. *Consider two clusters of size $M = \frac{n^{1/4}}{2c} \times \frac{n^{1/2}}{4}$ placed on the same horizontal line and separated by distance $\frac{n^{1/2}}{4}$. For a sufficiently large constant c, the maximum beamforming gain between the two given clusters can be achieved* *by using a proper compensation of the phase shifts at the transmit side, that is*

$$\left|\sum_{k=1}^{M} \exp(2\pi i(r_{jk} - x_k)/\lambda)\right| = \Theta(M),$$

where x_k denotes the horizontal position of node k.

Another building block for the multi-stage back-and-forth beamforming scheme is described in the following lemma.

Lemma III.4. *Consider two clusters of size $M = \sqrt{d}/c \times d$ placed on the same horizontal line and separated by distance d. Moreover, each node has a noisy observation of the same message (to be successfully decoded at the end by all the nodes) with an initial $\text{SNR} = \Theta(M^{-1})$. As such, for any $\epsilon > 0$, $P = O(d^{-1})$ and sufficiently large constant c, back-and-forth beamforming between the two clusters achieves a broadcast rate*

$$R_M = \Omega\left(\frac{M^{2-\epsilon} P}{d^2}\right) = \Omega(M^{\frac{2}{3}-\epsilon} P).$$

Proof. For the sake of clarity, we provide here a simplified proof, assuming that the distance from any node in the first cluster to any node in the second cluster is equal to the inter-cluster distance d. By the lemma III.3, a full beamforming gain is achievable. Practically, this allows us to assume that the phase shifts in the fading coefficients h_{jk} can be completely compensated. and replaced therefore by 0.

Before starting the back-and-forth beamforming between the two clusters, each of the nodes in the first cluster has a noisy observation of message X, given by

$$X_j = \sqrt{\text{SNR}}\ X + Z_j^{(0)} \qquad j = 1, \ldots, M$$

where $\mathbb{E}(|X|^2) = \mathbb{E}(|Z_j^{(0)}|^2) = \Theta(1)$. Note that it does not make a difference at which cluster the back-and-forth beamforming starts or ends. Hence, assume the nodes in the first cluster ignite the scheme by amplifying and forwarding the noisy observations of X to the nodes of the second cluster. The received signal at j-th node of the second cluster is given by

$$\begin{aligned} Y_j^{(1)} &= \frac{A}{d} \sum_{i=1}^{M} X_i + Z_j^{(1)} \\ &= \tfrac{A \times M}{d} \sqrt{\text{SNR}}\ X + \tfrac{A \times M}{d} \tfrac{1}{\sqrt{M}}\, Z^{(0)} + Z_j^{(1)} \end{aligned}$$

where $A = \sqrt{P/R_M}$ is the amplification factor[2], $Z^{(0)} = 1/\sqrt{M} \sum_{i=1}^{M} Z_i^{(0)}$ (note that $\mathbb{E}(|Z^{(0)}|^2) = \Theta(1)$), and $Z_j^{(1)}$ is additive white Gaussian noise of variance $\Theta(1)$. Repeating the same process k times in a back-and-forth manner results in a final signal at the j-th node of the first or the second cluster (depending on whether k is odd or even) that is given by

$$\begin{aligned} Y_j^{(k)} &= \left(\tfrac{A \times M}{d}\right)^k \sqrt{\text{SNR}}\ X + \left(\tfrac{A \times M}{d}\right)^k \tfrac{1}{\sqrt{M}}\, Z^{(0)} \\ &\quad + \ldots + \left(\tfrac{A \times M}{d}\right)^{k-l} \tfrac{1}{\sqrt{M}}\, Z^{(l)} + \ldots + Z_j^{(k)} \end{aligned}$$

[2]For a broadcast rate of R_M bits/slot, $1/R_M$ time slots per bit are used, resulting in a total transmit power of $A^2 = P/R_M$.

where $Z^{(l)} = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} Z_i^{(l)}$ for $l = 0, \ldots, k-1$ (again $\mathbb{E}(|Z^{(l)}|^2) = \Theta(1)$), and $Z_j^{(k)}$ is additive white Gaussian noise of variance $\Theta(1)$. We want the power of the signal

$$\mathbb{E}\left(\left(\left(\tfrac{A \times M}{d}\right)^k \sqrt{\text{SNR}}\ X\right)^2\right) = \left(\tfrac{A \times M}{d}\right)^{2k} \text{SNR} = \Theta(1)$$

$$\Rightarrow R_M = (d \times P)\ \text{SNR}^{\frac{1}{k}} = \Omega(d^{1-\epsilon} P) = \Omega\left(M^{\frac{2}{3}-\epsilon} P\right)$$

Finally, given the assumption that $\text{SNR} = \Theta(M^{-1})$, the power of the noise is indeed given by

$$\begin{aligned} &\sum_{l=0}^{k-1} \mathbb{E}\left(\left(\left(\tfrac{A \times M}{d}\right)^{k-l} \tfrac{1}{\sqrt{M}} Z^{(l)}\right)^2\right) + \mathbb{E}\left(\left(Z_j^{(k)}\right)^2\right) \\ &\leq k + 1 = \Theta(1) \end{aligned}$$

□

The proof of Theorem III.1 follows now directly from Lemmas III.2 and III.4

Proof of Theorem III.1. As shown in Fig 1, the network is divided into clusters of size $M = \frac{n^{1/4}}{2c} \times \frac{n^{1/2}}{4} = \Theta(n^{3/4})$. According to lemma III.2, there are order M nodes in each cluster with high probability. Clusters apart from each other by distance $n^{1/2}/4$ pair up to perform back-and-forth beamforming. Let us assume that at the beginning (i.e. after Phase 1) each node has a noisy observation of the same message with an $\text{SNR} = \Theta(M^{-1})$ (to be verified below). Moreover, as each cluster pair operates only a fraction of $\frac{1}{2cn^{1/4}}$ of the time, we can say that during the transmission, the power $\widehat{P}$ available at each node is equal to $2cn^{1/4}P$. As $P = O(n^{-3/4})$, it holds that $\widehat{P} = O(n^{-1/2})$. Applying Lemma III.4 with $d = n^{1/2}/4$, results in

$$R_M = \Omega\left(M^{\frac{2}{3}-\epsilon} \widehat{P}\right).$$

As such, multi-stage back-and-forth beamforming achieves a broadcast rate

$$R_n = \frac{R_M}{2cn^{1/4}} = \Omega\left(\frac{M^{\frac{2}{3}-\epsilon} P}{n^{\frac{1}{4}}}\right) = \Omega\left(n^{\frac{1}{2}-\epsilon} P\right).$$

Note finally that each source node operates every n/R_n time slots. It has therefore a transmit power of order $n^{1/2+\epsilon}$. Moreover, the distance separating any two nodes in the network is as most $\sqrt{2n}$. This implies that the SNR of the received signal at all the nodes in the network is $\Omega(n^{-1/2}) = \Omega(M^{-2/3})$. This is even better than the required SNR. □

IV. Optimality of the Scheme

In this section, we first establish a general upper bound on the broadcast capacity of wireless networks at low SNR, which applies to a general fading matrix H (with proper measurement units such that again, $\text{SNR}_s = P$ in these units).

Theorem IV.1. *Let us consider a network of n nodes and let H be the $n \times n$ matrix with $h_{jj} = 0$ on the diagonal and h_{jk} = the fading coefficient between node j and node k in* *the network. The broadcast capacity of such a network with n nodes is then bounded by*

$$C_n \leq P\,\|H\|^2$$

where P is the power available per node and $\|H\|$ is the spectral norm (i.e. the largest singular value) of H.

Proof. Using the classical cut-set bound, the following upper bound on the broadcast capacity C_n is obtained:

$$\begin{aligned} C_n &\leq \max_{\substack{p_X:\\ \mathbb{E}(|X_k|^2)\leq P,\ \forall 1\leq k\leq n}} \min_{1\leq j\leq n} I(X_{\{1,\ldots,n\}\setminus\{j\}};Y_j) \\ &\leq \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \min_{1\leq j\leq n} \log(1+h_j Q_X h_j^*) \end{aligned}$$

where $h_j = (h_{j1},\ldots,h_{j,j-1},0,h_{j,j+1},\ldots,h_{jn})$. Using the fact that the minimum of a set of numbers is less than its average, the above expression can be further bounded by

$$\begin{aligned} C_n &\leq \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \frac{1}{n}\sum_{j=1}^{n} \log(1+h_j Q_X h_j^*) \\ &= \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \frac{1}{n}\sum_{j=1}^{n} \log\det(I_n + h_j^* h_j Q_X) \\ &\leq \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \log\det\left(I_n + \frac{1}{n}\sum_{j=1}^{n} h_j^* h_j Q_X\right) \end{aligned}$$

using successively the property that $\log\det(I+AB) = \log\det(I+BA)$ and the fact that $\log\det(\cdot)$ is concave. Observing now that the $n\times n$ matrix H whose entries are given by $h_{jk} = (h_j)_k$ is the one of the theorem statement and that $\sum_{j=1}^n h_j^* h_j = H^*H$, we can rewrite, using again $\log\det(I+AB) = \log\det(I+BA)$:

$$\begin{aligned} C_n &\leq \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \log\det\left(I_n + \frac{1}{n} H Q_X H^*\right) \\ &\leq \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \frac{1}{n}\operatorname{Tr}(H Q_X H^*) \\ &\leq \max_{\substack{Q_X\geq 0\\ (Q_X)_{kk}\leq P,\ \forall 1\leq k\leq n}} \frac{1}{n}\operatorname{Tr}(Q_X)\,\|H\|^2 = P\,\|H\|^2 \end{aligned}$$

which completes the proof. □

We now aim to specialize Theorem IV.1 to line-of-sight fading, where the matrix H is given by

$$h_{jk} = \begin{cases} 0 & \text{if } j=k \\ \sqrt{G}\,\dfrac{\exp(2\pi i r_{jk})}{r_{jk}} & \text{if } j\neq k \end{cases} \qquad (2)$$

The rest of the present section is devoted to proving the proposition below which, together with Theorem IV.1, shows the asymptotic optimality of the back-and-forth beamforming scheme presented in Section III for two-dimensional networks at low SNR and under LOS fading[3].

[3]Note that in a similar context but in one-dimensional networks, Theorem IV.1 also allows to recover the result already obtained in [7].

Proposition IV.2. *Let H be the $n\times n$ matrix given by (2). For every $\varepsilon > 0$, there exists a constant $c>0$ such that*

$$\|H\|^2 \leq c\,n^{\frac{1}{2}+\varepsilon}$$

with high probability as n gets large.

Analyzing directly the the asymptotic behavior of $\|H\|$ reveals itself difficult. We therefore decompose our proof into simpler subproblems. The first building block of the proof is the following lemma, which can be viewed as a generalization of the classical Geršgorin discs' inequality.

Lemma IV.3. *Let B be an $n\times n$ matrix decomposed into blocks B_{jk}, $j,k=1,\ldots,K$, each of size $M\times M$, with $n = K\times M$. Then*

$$\|B\| \leq \max\left\{\max_{1\leq j\leq K}\sum_{k=1}^{K}\|B_{jk}\|,\ \max_{1\leq j\leq K}\sum_{k=1}^{K}\|B_{kj}\|\right\}$$

Due to space constraints, the proof of this lemma is relegated to the Appendix.

The second building block of this proof is the following claim. In Appendix A, we provide an approximation argument as well as some numerical simulations supporting this claim. The rigorous proof is still work in progress.

Claim IV.4. *Let H_0 be the $M\times M$ channel matrix between two square clusters of M nodes distributed uniformly at random, each of area $A = M$ and separated by distance d such that $\sqrt{M}\leq d\leq M$. Then there exists a constant $c>0$ such that*

$$\|H_0\|^2 \leq c\left(\frac{M}{d}\right)\log M$$

with high probability as M gets large.

The strategy for the proof of Proposition IV.2 is now essentially the following: in order to bound $\|H\|$, we divide the matrix into smaller blocks, apply Lemma IV.3 and use the estimate of Claim IV.4 in order to bound the off-diagonal terms $\|H_{jk}\|$. For the diagonal terms $\|H_{jj}\|$, we reapply Lemma IV.3 and proceed in a recursive manner, until we reach small size blocks for which a loose estimate is sufficient to conclude.

Let us therefore decompose the network into K clusters of M nodes each, with $n = K\times M$. By Lemma IV.3, we obtain

$$\|H\| \leq \max\left\{\max_{1\leq j\leq K}\sum_{k=1}^{K}\|H_{jk}\|,\ \max_{1\leq j\leq K}\sum_{k=1}^{K}\|H_{kj}\|\right\} \qquad (3)$$

where the $n\times n$ matrix H is decomposed into blocks H_{jk}, $j,k=1,\ldots,K$, with H_{jk} denoting the $M\times M$ channel matrix between cluster number j and cluster number k in the network. Let us also denote by d_{jk} the corresponding inter-cluster distance, measured from the centers of these clusters. According to Claim IV.4, if $d_{jk}\geq 2\sqrt{M}$, then there exists a constant $c>0$ such that

$$\|H_{jk}\|^2 \leq c\left(\frac{M}{d_{jk}}\right)\log M \leq c\left(\frac{M}{d_{jk}}\right)\log n$$

with high probability as $M\to\infty$.

Let us now fix $j \in \{1, \dots, K\}$ and define $R_j = \{1 \le k \le K : d_{jk} < 2\sqrt{M}\}$ and $S_j = \{1 \le k \le K : d_{jk} \ge 2\sqrt{M}\}$. By the above inequality, we obtain

$$\sum_{k=1}^{K} \|H_{jk}\| \le \sum_{k \in R_j} \|H_{jk}\| + \sqrt{c \log n} \sum_{k \in S_j} \sqrt{\frac{M}{d_{jk}}}$$

with high probability as M gets large. Observe that as there are order l clusters at distance $l\sqrt{M}$ from cluster j, we obtain

$$\sum_{k \in S_j} \sqrt{\frac{M}{d_{jk}}} \simeq \sum_{l=1}^{\sqrt{K}} l \sqrt{\frac{M}{l\sqrt{M}}} \simeq M^{1/4} K^{3/4} = \frac{n^{3/4}}{M^{1/2}}$$

as $K = n/M$. There remains to upperbound the sum over R_j. Observe that this sum contains at most 9 terms: namely the term $k = j$ and the 8 terms corresponding to the 8 neighboring clusters of cluster j. It should then be observed that for each $k \in R_j$, $\|H_{jk}\| \le \|H(R_j)\|$, where $H(R_j)$ is the $9M \times 9M$ matrix made of the 9×9 blocks H_{i_1,i_2} such that $i_1, i_2 \in R_j$. Finally, this leads to

$$\sum_{k=1}^{K} \|H_{jk}\| \le 9\|H(R_j)\| + \sqrt{c \log n}\, \frac{n^{3/4}}{M^{1/2}}$$

Using the symmetry of this bound and (3), we obtain

$$\|H\| \le 9 \max_{1 \le j \le K} \|H(R_j)\| + \sqrt{c \log n}\, \frac{n^{3/4}}{M^{1/2}} \qquad (4)$$

A key observation is now the following: the $9M \times 9M$ matrix $H(R_j)$ has exactly the same structure as the original matrix H. So in order to bound its norm $\|H(R_j)\|$, the same technique may be reused! This leads to the following recursive lemma.

Lemma IV.5. *Assume there exist constants $c > 0$ and $b \in [1/4, 1/2]$ such that*

$$\|H\| \le \sqrt{c \log n}\, n^b$$

with high probability as n gets large. Then there exists a constant $c' > 0$ such that

$$\|H\| \le \sqrt{c' \log n}\, n^{\frac{3b}{4b+2}}$$

with high probability as n gets large.

Proof. The assumption made implies that there exist $c > 0$ and $b \in [1/4, 1/2]$ such that for every $M \times M$ diagonal subblock H_M of the matrix H,

$$\|H_M\| \le \sqrt{c \log M}\, M^b$$

with high probability as M gets large. Together with (4), this implies that

$$\|H\| \le 9\sqrt{c \log M}\, M^b + \sqrt{c \log n}\, \frac{n^{3/4}}{M^{1/2}} \le \sqrt{c \log n} \left(M^b + \frac{n^{3/4}}{M^{1/2}} \right)$$

Choosing $M = \lceil n^{3/(4b+2)} \rceil$, we obtain

$$\|H\| \le \sqrt{c' \log n}\, n^{3b/(4b+2)}$$

□

It is now easy to check that the assumption of Lemma IV.5 holds with $b = 1/2$. Apply for this the slighlty modified version of the classical Geršgorin inequality (which is nothing but the statement of Lemma IV.3 applied to the case $M = 1$):

$$\|H\| \le \max\left\{ \max_{1 \le j \le n} \sum_{k=1}^{n} |h_{jk}|, \max_{1 \le j \le n} \sum_{k=1}^{n} |h_{kj}| \right\}$$

Using (2), we see that

$$\sum_{k=1}^{n} |h_{jk}| = \sum_{k=1}^{n} \frac{1}{r_{jk}} \simeq \sum_{l=1}^{\sqrt{n}} l\, \frac{1}{l} = \sqrt{n}$$

which implies that $\|H\| \le \sqrt{n}$.

By applying Lemma IV.5 successively, we obtain a decreasing sequence of upper bounds on $\|H\|$:

$$\|H\| \le \sqrt{c \log n}\, n^{b_0}, \quad \le \sqrt{C \log n}\, n^{b_1}, \quad \le \sqrt{c \log n}\, n^{b_2}$$

where the sequence $b_0 = 1/2$, $b_1 = 3b_0/(4b_0 + 2) = 3/8$, $b_2 = 3b_1/(4b_1 + 2) = 9/28$ converges to the fixed point $b^* = 1/4$. This finally proves Proposition IV.2 (modulo Claim IV.4). □

V. Conclusion

In this work, we characterize the broadcast capacity of two-dimensional wireless networks at low SNR in line-of-sight environment, which is achieved via a back-and-forth beamforming scheme. A possible simplification of the scheme is to reduce the time-division factor so as to have back-and-forth transmissions directly between the two halves of the network. The analysis of the performance of this simplified scheme is however more delicate.

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APPENDIX

Proof of Lemma III.2. The number of nodes in a given cluster is the sum of n independently and identically distributed Bernoulli random variables B_i, with $\mathcal{P}(B_i = 1) = M/n$. Hence

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{n} B_i \geq (1+\delta)M\right) \\
&= \mathbb{P}\left(\exp\left(s\sum_{i=1}^{n} B_i\right) \geq \exp(s(1+\delta)M)\right) \\
&\leq \mathbb{E}^n(\exp(sB_1))\exp(-s(1+\delta)M) \\
&= \left(\frac{M}{n}\exp(s) + 1 - \frac{M}{n}\right)^n \exp(-s(1+\delta)M) \\
&\leq \exp(-M(s(1+\delta) - \exp(s) + 1)) = \exp(-M\Delta_+(\delta))
\end{aligned}$$

where $\Delta_+(\delta) = (1+\delta)\log(1+\delta) - \delta$ by choosing $s = \log(1+\delta)$. The proof of the lower bound follows similarly by considering the random variables $-B_i$. The conclusion follows from the union bound. □

Proof of Lemma III.3. The distance between transmit node k and receive node j is given by

$$r_{jk} = \sqrt{(\sqrt{n}/4 + x_j + x_k)^2 + (y_j - y_k)^2)},$$

as shown in Fig. 2, where $0 \leq x_k \leq \sqrt{n}/4$ (resp. x_j) and $0 \leq y_k \leq n^{1/4}/2c$ (resp. y_j) denote the horizonal and the vertical coordinates of the node k (resp. node j). Moreover, the two clusters are separated by distance $\sqrt{n}/4$. It is clear

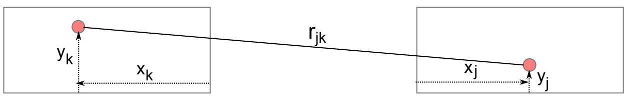

Fig. 2.

that $r_{jk} \geq \sqrt{n}/4 + x_j + x_k$. Moreover, we can upper bound r_{jk} as follows:

$$\begin{aligned}
r_{jk} &= \sqrt{(\sqrt{n}/4 + x_j + x_k)^2 + (y_j - y_k)^2)} \\
&\leq \sqrt{n}/4 + x_j + x_k + \frac{2(y_j - y_k)^2}{\sqrt{n}} \\
&\leq \sqrt{n}/4 + x_j + x_k + \frac{1}{2c^2}.
\end{aligned}$$

Therefore, by removing the horizontal coordinates at the transmit nodes and for sufficiently large constant c, we obtain

$$\begin{aligned}
&\left|\sum_{k=1}^{M} \exp(2\pi i(r_{jk} - x_k)/\lambda)\right| \\
=&\left|e^{(2\pi i(\sqrt{n}/4 + x_j)}\sum_{k=1}^{M} e^{(2\pi i(r_{jk} - (\sqrt{n}/4 + x_j) - x_k})/\lambda)\right| \\
=&\left|\sum_{k=1}^{M} \exp(2\pi i(r_{jk} - \sqrt{n}/4 - x_j - x_k)/\lambda)\right| \\
\geq&\text{Re}\left(\sum_{k=1}^{M} \exp(2\pi i(r_{jk} - \sqrt{n}/4 - x_j - x_k)/\lambda)\right) \\
=&\sum_{k=1}^{M} \cos\left(2\pi i(r_{jk} - \sqrt{n}/4 - x_j - x_k)/\lambda)\right) \\
\geq&M \times \cos\left(\frac{2\pi i}{2\lambda c^2}\right) = \Theta(M).
\end{aligned}$$

□

Proof of Lemma IV.3. - Let us first consider the case where B is a Hermitian and positive semi-definite matrix. Then $\|B\| = \lambda_{\max}(B)$, the largest eigenvalue of B. Let now λ be an eigenvalue of B and u be its corresponding eigenvector, so that $\lambda u = Bu$. Using the block representation of the matrix B, we have

$$\lambda\, u_j = \sum_{k=1}^{K} B_{jk}\, u_k, \quad \forall 1 \leq j \leq K$$

where u_j is the j^{th} block of the vector u. Let now j be such that $\|u_j\| = \max_{1\leq k\leq K}\|u_k\|$. Taking norms and using the triangle inequality, we obtain

$$\begin{aligned}
|\lambda|\,\|u_j\| = \left\|\sum_{k=1}^{K} B_{jk}\, u_k\right\| &\leq \sum_{k=1}^{K}\|B_{jk}\, u_k\| \\
&\leq \sum_{k=1}^{K}\|B_{jk}\|\,\|u_k\| \leq \sum_{k=1}^{K}\|B_{jk}\|\,\|u_j\|
\end{aligned}$$

by the assumption made above. As $u \not\equiv 0$, $\|u_j\| > 0$, so we obtain

$$|\lambda| \leq \max_{1\leq j\leq K}\sum_{k=1}^{K}\|B_{jk}\|$$

As this inequality applies to any eigenvalue λ of B and $\|B\| = \lambda_{\max}(B)$, the claim is proved in this case.

- In the general case, observe first that $\|B\|^2 = \lambda_{\max}(BB^*)$, where BB^* is Hermitian and positive semi-definite. So by what was just proved above,

$$\|B\|^2 = \lambda_{\max}(BB^*) \leq \max_{1\leq j\leq K}\sum_{k=1}^{K}\|(BB^*)_{jk}\|$$

Now, $(BB^*)_{jk} = \sum_{l=1}^K B_{jl} B_{kl}^*$ so

$$\sum_{k=1}^K \|(BB^*)_{jk}\| = \sum_{k=1}^K \left\| \sum_{l=1}^K B_{jl} B_{kl}^* \right\|$$
$$\leq \sum_{k=1}^K \sum_{l=1}^K \|B_{jl}\| \, \|B_{kl}\| \leq \sum_{l=1}^K \|B_{jl}\| \max_{1 \leq j \leq K} \sum_{k=1}^K \|B_{kj}\|$$

and we finally obtain

$$\|B\|^2 \leq \left(\max_{1 \leq j \leq K} \sum_{l=1}^K \|B_{jl}\| \right) \left(\max_{1 \leq j \leq K} \sum_{k=1}^K \|B_{kj}\| \right)$$

which implies the result, as $ab \leq \max\{a,b\}^2$ for any two positive numbers a, b. $\square$

A. Arguments supporting Claim IV.4

The strategy we propose in order to upperbound $\|H_0\|^2$ is to use the moments' method, relying on the following inequality:

$$\|H_0\|^2 = \lambda_{\max}(H_0 H_0^*) \leq \left(\sum_{k=1}^M (\lambda_k(H_0 H_0^*))^\ell \right)^{1/\ell}$$
$$= \left(\mathrm{Tr}\left((H_0 H_0^*)^\ell \right) \right)^{1/\ell}$$

valid for any $\ell \geq 1$. So by Jensen's inequality, we obtain that $\mathbb{E}(\|H_0\|^2) \leq \left(\mathbb{E}(\mathrm{Tr}((H_0 H_0^*)^\ell)) \right)^{1/\ell}$. An approximation argument similar to that used in [10] suggests that the following two matrices have similar spectral properties:

$$(H_0)_{jk} = \frac{\exp(2\pi i r_{jk})}{r_{jk}} \quad \text{and} \quad (H_0')_{jk} = \frac{\exp(2\pi i m y_j z_k)}{d}$$

where $m = M/d$ and y_j, z_k are i.i.d.$\sim \mathcal{U}([0,1])$ random variables. This suggests that

$$\mathbb{E}(\mathrm{Tr}((H_0 H_0^*)^\ell)) \simeq \mathbb{E}(\mathrm{Tr}((H_0' H_0'^*)^\ell))$$

In addition, it can be shown, following the technique developed in [10], that there exists a constant $c > 0$ such that

$$\mathbb{E}(\mathrm{Tr}((H_0' H_0'^*)^\ell)) \leq \left(c \, \frac{M}{d} \, \log M \right)^{\ell+1} \quad \text{for } \sqrt{M} \leq d \leq M$$

A slightly stronger estimate than the above is actually confirmed for H_0 itself through numerical simulations: see Figure 3 on the next page.

Taking finally $\ell \to \infty$ leads to $\mathbb{E}(\|H_0\|^2) \leq c \, \frac{M}{d} \, \log M$ (modulo the approximation made). Besides, using a standard concentration argument, one can prove that for every $\varepsilon > 0$, there exists a constant $c > 0$ such that

$$\mathbb{P}\left(\left| \|H_0\|^2 - \mathbb{E}(\|H_0\|^2) \right| > c \, M^{\frac{1}{2}+\varepsilon} \right) \underset{M \to \infty}{\to} 0$$

which completes the argument.

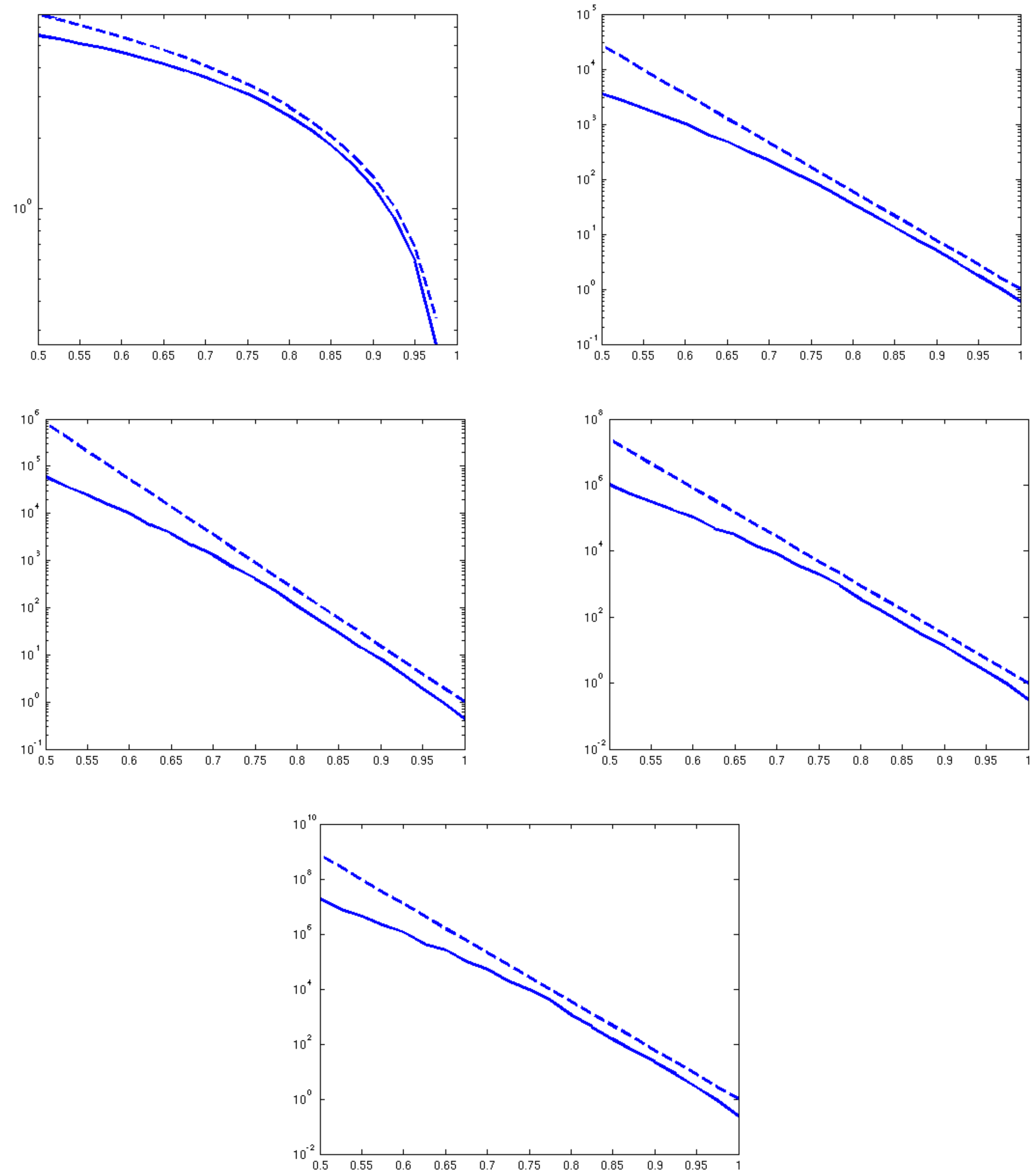

Fig. 3. Simultaneous representation of $\mathbb{E}(\mathrm{Tr}((H_0 H_0^*)^\ell))$ (solid curve) and $(M/d)^{\ell+1}$ (dotted curve) for the successive values of $\ell = 1, 2, 3, 4, 5$. On the graphs, $M = 900$ and $d = M^\alpha$ with $\alpha \in [0.5, 1]$, whose value is represented on the horizontal axis, while the vertical scale is logarithmic.