# On the Impact of Sparsity on the Broadcast Capacity of Wireless Networks

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***Abstract*—We characterize the maximum achievable broadcast rate in a wireless network at low SNR and under line-of-sight fading assumption. Our result shows that this rate depends negatively on the sparsity of the network. This is to be put in contrast with the number of degrees of freedom available in the network, which have been shown previously to increase with the sparsity of the network.**

***Index Terms*—wireless networks, broadcast capacity, low SNR communications, beamforming strategies, random matrices**

## I. Introduction

There is a vast body of literature on the subject of multiple-unicast communications in ad hoc wireless networks. Because of the inherent broadcast nature of wireless signals, managing the *interference* between the multiple source-destination pairs is a key issue and has led to various interesting proposals [1], [2], [3], [4]. In some of these works, it appeared that the model considered for the fading environment may substantially impact the performance of the proposed communication schemes (see [5]). In particular, the *channel diversity*, both spatial and temporal, turns out to be a key parameter for the analysis of the various schemes.

In the present paper, we address an a priori much easier scenario (previously considered in [6], [7]). Instead of every source node willing to communicate each to a different destination node, we consider the *broadcast scenario*, where each source node wishes to send some piece of information to all the other nodes in the network. This situation is to be encountered e.g. when control signals carrying channel state information should be broadcasted to the whole network. In this context, the broadcast nature of the wireless medium can only help relaying communications, so that the situation seems simpler to handle, if not trivial. What we show in the following is that even in this simpler scenario, the optimal communication performance highly depends on the nature of the wireless medium. The conclusions we draw put again channel diversity to the forefront. But whereas diversity was beneficial for establishing multiple parallel communication channels in the multiple-unicast scenario, it turns out that in the present case, diversity is on the contrary detrimental to a proper broadcasting of information. A *duality* is further established between the number of *degrees of freedom* available for multi-party communications and the *beamforming gain* of broadcast transmissions, which allows for a better dissemination of information. At one end, in a rich scattering environment, degrees of freedom are prominent, while beamforming is practically infeasible. At the other end, degrees of freedom become a scarce resource, while high beamforming gains can be achieved via collaborative transmissions.

Our analysis relies on the simplistic line-of-sight fading model for signal attenuation over distance, where signal amplitude attenuation is inversely proportional to distance and phase shifts are also proportional to distance. Yet, this model, along with another parameter characterizing the *sparsity* of the network, allows to capture the different regimes mentioned above and to characterize the performance trade-offs. In addition, we would like to highlight here that despite the simplicity of the model, the mathematical analysis needed to establish the result on the maximum achievable broadcast rate in the network requires a precise and careful study of the spectral norm of unconventional random matrices, rarely studied in the mathematical literature.

## II. Model

There are $n$ nodes uniformly and independently distributed in a square of area $A = n^\nu$, $\nu > 0$. Every node wants to broadcast a different message to the whole network, and all nodes want to communicate at a common *per user* data rate $r_n$ bits/s/Hz. We denote by $R_n = n\, r_n$ the resulting *aggregate* data rate and will often refer to it simply as "broadcast rate" in the sequel. The broadcast capacity of the network, denoted as $C_n$, is defined as the maximum achievable aggregate data rate $R_n$. We assume that communication takes place over a flat fading channel with bandwidth $W$ and that the signal $Y_j[m]$ received by the $j$-th node at time $m$ is given by

$$Y_j[m] = \sum_{k \in \mathcal{T}} h_{jk}\, X_k[m] + Z_j[m],$$

where $\mathcal{T}$ is the set of transmitting nodes, $X_k[m]$ is the signal sent at time $m$ by node $k$ and $Z_j[m]$ is additive white circularly symmetric Gaussian noise (AWGN) of power spectral density $N_0/2$ Watts/Hz. We also assume a common average power budget per node of $P$ Watts, which implies that the signal $X_k$ sent by node $k$ is subject to an average power constraint $\mathbb{E}(|X_k|^2) \leq P$. In line-of-sight environment, the complex baseband-equivalent channel gain $h_{jk}$ between transmit node $k$ and receive node $j$ is given by

$$h_{jk} = \sqrt{G}\, \frac{\exp(2\pi i r_{jk}/\lambda)}{r_{jk}}, \tag{1}$$

where $G$ is Friis' constant, $\lambda$ is the carrier wavelength, and $r_{jk}$ is the distance between node $k$ and node $j$. Let us finally define

$$\mathrm{SNR}_s = \frac{GP}{N_0 W} n^{1-\nu},$$

which is the SNR available for a communication between two (neighboring) nodes at distance $n^{\frac{\nu-1}{2}}$ in the network.

We focus in the following on the low SNR regime, by which we mean that $\mathrm{SNR}_s = n^{-\gamma}$ for some constant $\gamma > 0$. This means that the power available at each node does not allow for a constant rate direct communication with a neighbor. This could be the case e.g., in a sensor network with low battery nodes, or in a sparse network (large $\nu$) with long distances between neighboring nodes.

In order to simplify notation, we choose new measurement units such that $\lambda = 1$ and $G/(N_0 W) = 1$ in these units. This allows us to write in particular that $\mathrm{SNR}_s = n^{1-\nu} P$.

## III. Main result

Our main result is the following: the maximum achievable aggregate broadcast rate scales as[1]

$$R_n \sim \begin{cases} \min\{\mathrm{SNR}_s, 1\} & \text{if } (A/\lambda^2) \geq n^2 \\ \min\left\{ \left( \dfrac{n}{\sqrt{A}/\lambda} \right) \mathrm{SNR}_s, 1 \right\} & \text{if } 1 \leq (A/\lambda^2) \leq n^2 \end{cases}$$

and is achieved by a simple broadcast transmission in the first case and by a multi-stage beamforming strategy in the second case, described in length in our previous paper [8]. The performance is further capped at 1, which means that such beamforming gains can only be obtained at low SNR.

We see here for a sparse network of density $O(1/n)$ (regime where $A \sim n^2$), no particular beamforming gain can be obtained, while the beamforming gain increases as the network gets denser and denser. Let us mention here that the result where the network is of constant density ($A \sim n$) has been previously established in [8]. The extension is performed here for all possible scalings of the area $A$ with respect to the number of nodes $n$. Establishing the capacity upper bound is highly non-trivial in this more general setup.

*Duality between the broadcast capacity and the number of degrees of freedom.* Let us recall what is known for the multiple-unicast scenario [9]. In this case, the aggregated network throughput scales as

$$T_n \sim \begin{cases} n\, \mathrm{SNR}_s & \text{if } (A/\lambda^2) \geq n^2 \\ (\sqrt{A}/\lambda)\, \mathrm{SNR}_s & \text{if } n \leq (A/\lambda^2) \leq n^2 \\ \sqrt{n}\, \mathrm{SNR}_s & \text{if } 1 \leq A/\lambda^2 \leq n \end{cases}.$$

Such an aggregate throughput is achieved by a hierarchical cooperative strategy involving network-wide distributed MIMO transmissions in the first two cases, while simple multi-hopping achieves the performance claimed in the third case.

[1] as $n$ gets large, up to logarithmic factors. The result here is stated to highlight the dependency of the broadcast rate $R_n$ on the sparsity of the network through the term $n/\sqrt{A}$ and on the SNR at the nearest neighboring node through the parameter $\mathrm{SNR}_s$. Note that both $\mathrm{SNR}_s$ and $A$ depend on the parameter $\nu$ defined in the previous section.

We therefore see that the wider the area is, the more degrees of freedom are available for communication in the network. The regime where $A \sim n^2$ (corresponding to a sparse network of density $O(1/n)$) models the case where the phase shifts are large enough to ensure sufficient channel diversity and full degrees of freedom of MIMO transmissions. On the contrary, in the regime where $A \sim n$ (corresponding to a network of constant density), phase shifts do not allow for efficient MIMO transmissions, so that multi-hopping becomes the best way to transfer information across the network.

We can finally observe the duality of the two results: in the regime where $A/\lambda^2 \geq n$ (that is, for networks of constant density or sparser) and at low SNR, we have

$$\frac{T_n}{\mathrm{SNR}_s} \frac{R_n}{\mathrm{SNR}_s} = n$$

which captures the fact that high beamforming gains can only be obtained at the expense of a reduced number of degrees of freedom (or reciprocally).

## IV. Broadcasting Strategies in Different Regimes

First note that under the LOS model (1) and the assumptions made in the Section II, a simple time division scheme achieves a broadcast (aggregate) rate $R_n$ of order $\min(\mathrm{SNR}_s, 1)$. Indeed, a rate of order 1 is obviously achieved at high SNR[2]. At low SNR (i.e. when $\mathrm{SNR}_s \sim n^{-\gamma}$ for some $\gamma > 0$), each node can spare power while the others are transmitting, so as to compensate for the path loss of order $1/n^\nu$ between the source node and other nodes located at distance at most $\sqrt{2n^\nu}$, leading to a broadcast rate of order $R_n \sim \log(1 + nP/n^\nu) \sim n^{1-\nu} P = \mathrm{SNR}_s$.

In the following, we will see that, at low SNR, while the described simple TDMA based broadcast scheme is order-optimal for $A \geq n^2$, it is not optimal for sparse networks with area $A < n^2$ ($\nu < 2$) (for simplicity, as stated in Section II, we take $\lambda = 1$). On the other hand, the back-and-forth beamforming scheme, presented in [7], [8], proves to be order-optimal for $A \leq n^2$.

As described in [8], the back-and-forth beamforming scheme involves source nodes taking turns to broadcast their messages. Each transmission is followed by a series of network-wide back-and-forth transmissions that reinforce the strength of the signal, so that at the end, every node is able to decode the message sent from the source. The reason why back-and-forth transmissions are useful for **small area networks/dense networks** is that in line-of-sight environment, nodes are able to (partly) align the transmitted signals so as to create a significant beamforming gain for each transmission (whereas this would not be the case in **high scattering environment/sparse networks** with i.i.d. fading coefficients). In short, the back-and-forth beamforming scheme is split into two phases:

**Phase 1. Broadcast Transmission.** The source node broadcasts its message to the whole network. All the

[2] We coarsely approximate $\log P$ by 1 here!

nodes receive a noisy version of the signal, which remains undecoded. This phase only requires one time slot.

**Phase 2. Back-and-Forth Beamforming with Time Division.** Upon receiving the signal from the broadcasting node, nodes start multiple back-and-forth beamforming transmissions between the two halves of the network to enhance the strength of the signal. Although this simple scheme probably achieves the optimal performance claimed in Theorem IV.1 below, we lack the analytical tools to prove it. For this reason, we propose a time-division strategy, where clusters of size $M = \frac{n^{\nu/4}}{2c_1} \times \frac{n^{\nu/2}}{4}$ and separated by horizontal distance $d = \frac{n^{\nu/2}}{4}$ pair up for the back-and-forth transmissions. During each transmission, there are $\Theta\left(n^{\nu/4-\epsilon}\right)$ cluster pairs operating in parallel, so $\Theta(n^{1-\epsilon})$ nodes are communicating in total. The number of rounds needed to serve all nodes must therefore be $\Theta(n^{\epsilon})$.

After each transmission, the signal received by a node in a given cluster is the sum of the signals coming from the facing cluster, of those coming from other clusters, and of the noise. We assume a sufficiently large vertical distance $c_2 n^{\nu/4+\epsilon}$ separating any two cluster pairs. We show below that the broadcast rate between the operating clusters is $\Theta(n^{2-\frac{3\nu}{2}}P) = \Theta(n^{1-\frac{\nu}{2}}\mathrm{SNR}_s)$. Since we only need $\Theta(n^{\epsilon})$ number of rounds to serve all clusters, phase 2 requires $\Theta(n^{-2+\frac{3\nu}{2}+\epsilon}P^{-1})$ time slots. As such, back-and-forth beamforming achieves a broadcast rate of $\Theta(n^{2-\frac{3\nu}{2}-\epsilon}P) = \Theta(n^{1-\frac{\nu}{2}-\epsilon}\mathrm{SNR}_s)$ bits per time slot. In view of the described scheme, we are able to state the following result.

**Theorem IV.1.** *For any $\epsilon > 0$, $0 < \nu < 2$, and $P = O(n^{-2+\frac{3\nu}{2}})$, the following broadcast rate*

$$R_n = \Omega\left(n^{2-\frac{3\nu}{2}-\epsilon}P\right) = \Omega\left(n^{1-\frac{\nu}{2}-\epsilon}\mathrm{SNR}_s\right)$$

*is achievable with high probability*[3] *in the network.*

Before proceeding with the proof of the theorem, the following lemma provides an upper bound on the probability that the number of nodes inside each cluster deviates from its mean by a large factor. The proof can be found in [10].

**Lemma IV.2.** *Let us consider a cluster of area $M$ with $M = n^{\beta}$ for some $\nu - 1 < \beta < \nu$. The number of nodes inside each cluster is then between $((1-\delta)Mn^{1-\nu}, (1+\delta)Mn^{1-\nu})$ with probability larger than $1 - \frac{n^{\nu}}{M}\exp(-\Delta(\delta)Mn^{1-\nu})$ where $\Delta(\delta)$ is independent of $n$ and satisfies $\Delta(\delta) > 0$ for $\delta > 0$.*

Two clusters of size $M = \frac{n^{\nu/4}}{2c_1} \times \frac{n^{\nu/2}}{4}$ placed on the same horizontal line and separated by distance $d = n^{\nu/2}/4$ form a cluster pair. During the back-and-forth beamforming phase, there are many cluster pairs operating simultaneously. Given that the cluster width is $\frac{n^{\nu/4}}{2c_1}$ and the vertical separation between adjacent cluster pairs is $c_2 n^{\nu/4+\epsilon}$, there are $N_C = \Theta\left(n^{\nu/4-\epsilon}\right)$ cluster pairs operating at the same time. Let $\mathcal{R}_i$ and $\mathcal{T}_i$ denote the receiving and the transmitting clusters of the $i$-th cluster pair, respectively.

*Proof of Theorem IV.1.* The first phase of the scheme results in noisy observations of the message $X$ at all nodes, which are given by

$$Y_k^{(0)} = \sqrt{\mathrm{SNR}_k}\, X + Z_k^{(0)},$$

where $\mathbb{E}(|X|^2) = \mathbb{E}(|Z_k^{(0)}|^2) = 1$ and $\mathrm{SNR}_k$ is the signal-to-noise ratio of the signal $Y_k^{(0)}$ received at the $k$-th node. In what follows, we drop the index $k$ from $\mathrm{SNR}_k$ and only write $\mathrm{SNR} = \min_k\{\mathrm{SNR}_k\}$. Note that it does not make a difference at which side of the cluster pairs the back-and-forth beamforming starts or ends. Hence, assume the left-hand side clusters ignite the scheme by amplifying and forwarding the noisy observations of $X$ to the right-hand side clusters. The signal received at node $j \in \mathcal{R}_i$ is given by

$$Y_j^{(1)} = \sum_{l=1}^{N_C} \sum_{k\in\mathcal{T}_l} \frac{\exp(2\pi i(r_{jk} - x_k))}{r_{jk}} A Y_k^{(0)} + Z_j^{(1)} \qquad (2)$$

where $A$ is the amplification factor (to be calculated later) and $Z_j^{(1)}$ is additive white Gaussian noise of variance $\Theta(1)$. By applying Lemma III.3 and Lemma III.4 in [10], we get

$$\left|\sum_{l=1}^{N_C} \sum_{k\in\mathcal{T}_l} \frac{\exp(2\pi i(r_{jk} - x_k))}{r_{jk}}\right| = \Theta\left(\frac{Mn^{1-\nu}}{d}\right),$$

where $x_k$ denotes the horizontal position of node $k$. For the sake of clarity, we can therefore approximate (to make readable) the expression in (2) as follows

$$Y_j^{(1)} \simeq \frac{AMn^{1-\nu}}{d}\sqrt{\mathrm{SNR}}\, X + \frac{A\sqrt{N_C M n^{1-\nu}}}{d} Z^{(0)} + Z_j^{(1)},$$

where

$$Z^{(0)} = \frac{d}{\sqrt{N_C M n^{1-\nu}}} \sum_{l=1}^{N_C} \sum_{k\in\mathcal{T}_l} \frac{\exp(2\pi i(r_{jk} - x_k))}{r_{jk}} Z_k^{(0)}.$$

Note that $\mathbb{E}(|Z^{(0)}|^2) = \Theta(1)$. Repeating the same process $t$ times in a back-and-forth manner results in a final signal at node $j \in \mathcal{R}_i$ given by

$$Y_j^{(k)} = \left(\frac{AMn^{1-\nu}}{d}\right)^t \sqrt{\mathrm{SNR}}\, X + \left(\frac{AMn^{1-\nu}}{d}\right)^t \sqrt{\frac{N_C}{Mn^{1-\nu}}}\, Z^{(0)} + \ldots + \left(\frac{AMn^{1-\nu}}{d}\right)^{t-s} \sqrt{\frac{N_C}{Mn^{1-\nu}}}\, Z^{(s)} + \ldots + Z_j^{(t)},$$

where

$$Z^{(s)} = \frac{d}{\sqrt{N_C M n^{1-\nu}}} \sum_{b=1}^{N_C} \sum_{k\in\mathcal{T}_b} \frac{\exp(2\pi i(r_{jk} - x_k))}{r_{jk}} Z_k^{(s)}.$$

Note again that $\mathbb{E}(|Z^{(s)}|^2) = \Theta(1)$, and $Z_j^{(t)}$ is additive white Gaussian noise of variance $\Theta(1)$. We want the power of the signal to be of order of 1, which results in

$$A = \Theta\left(\frac{d}{Mn^{1-\nu}}\,\mathrm{SNR}^{-\frac{1}{2t}}\right). \qquad (3)$$

[3]that is, with probability at least $1 - O\left(\frac{1}{n^p}\right)$ as $n \to \infty$, where the exponent $p$ is as large as we want.

Since at each round of TDMA cycle there are $\Theta\left(N_C M n^{1-\nu}\right) = \Theta\left(n^{1-\epsilon}\right)$ nodes transmitting, then every node will be active $\Theta\left(\frac{N_C M n^{1-\nu}}{n}\right)$ fraction of the time. As such, the amplification factor is given by

$$A = \Theta\left(\sqrt{\frac{n^\nu}{N_C M}\tau P}\right), \quad (4)$$

where $\tau$ is the number of time slots between two consecutive transmissions. Equating (3) and (4), results in $\tau = O\left(\frac{1}{P}\left(\frac{d}{Mn^{1-\nu}}\right)^2 n^{-\epsilon}\mathrm{SNR}^{-1/t}\right)$. We can pick the number of back-and-forth transmissions $t$ sufficiently large to ensure that $\mathrm{SNR}^{-\frac{1}{t}} = O(n^\epsilon)$, which results in

$$\tau = O\left(\frac{1}{P}\left(\frac{d}{Mn^{1-\nu}}\right)^2\right) = O\left(\frac{1}{n^{2-\frac{3\nu}{2}}P}\right).$$

Given the required $\tau = O\left(\frac{1}{n^{2-3\nu/2}P}\right)$, one can check that the noise power is of order of 1. Moreover, we can see that for $P = O(n^{3\nu/2-2})$ the broadcast rate between simultaneously operating clusters is $\Omega(n^{2-3\nu/2}P)$. Finally, applying TDMA of $\frac{n}{N_C M n^{1-\nu}} = \Theta(n^\epsilon)$ steps ensures that $X$ is successfully decoded at all nodes and the broadcast rate $R_n = \Omega\left(n^{2-3\nu/2-\epsilon}P\right)$. This concludes the proof. □

## V. Optimality of the Scheme

We start with the general upper bound already established in [7] on the broadcast capacity of wireless networks at low SNR, which applies to a general fading matrix $H$.

**Theorem V.1.** *Let us consider a network of $n$ nodes and let $H$ be the $n \times n$ matrix with $h_{jj} = 0$ on the diagonal and $h_{jk}$ = the fading coefficient between node $j$ and node $k$ in the network. The broadcast capacity of such a network with $n$ nodes is then upper bounded by*

$$C_n \le P\,\|H\|^2$$

*where $P$ is the power available per node and $\|H\|$ is the spectral norm (i.e. the largest singular value) of $H$.*

We now aim to specialize Theorem V.1 to line-of-sight fading, where the matrix $H$ is given by

$$h_{jk} = \begin{cases} 0 & \text{if } j = k \\ \dfrac{\exp(2\pi i r_{jk})}{r_{jk}} & \text{if } j \ne k \end{cases} \quad (5)$$

The rest of the section is devoted to proving the proposition below which, together with Theorem V.1, shows the asymptotic optimality of the back-and-forth beamforming scheme for **small area networks/dense networks** ($0 < \nu < 2$) and the asymptotic optimality of the simple TDMA based broadcast scheme for **high scattering environment/sparse networks** ($\nu \ge 2$) at low SNR and under LOS fading.

**Proposition V.2.** *Let $H$ be the $n \times n$ matrix given by* (5). *For every $\varepsilon > 0$, there exists a constant $c > 0$ such that*

$$\|H\|^2 \le \begin{cases} c\,n^{2-\frac{3\nu}{2}+\epsilon}P = c\,n^{1-\frac{\nu}{2}+\epsilon}\ \mathrm{SNR}_s & \text{if } 0 < \nu < 2 \\ c\,n^{1-\nu+\epsilon}P = c\,n^\epsilon\ \mathrm{SNR}_s & \text{if } \nu \ge 2 \end{cases}$$

*with high probability as $n$ gets large.*

Analyzing directly the asymptotic behavior of $\|H\|$ reveals itself difficult. We therefore decompose our proof into simpler subproblems. The first building block of the proof is the following lemma, the proof of which is given in [10].

**Lemma V.3.** *Let $\widehat{H}$ be the $m\times m$ channel matrix between two square clusters of $m$ nodes distributed uniformly at random, each of area $A = m^\nu$, $\nu > 0$, then*

$$\|\widehat{H}\|^2 \le \begin{cases} \frac{m^{2+\epsilon}}{Ad} & \text{if } 0 < \nu < 2 \\ \max\left\{\frac{m^{2+\epsilon}}{Ad}, \frac{m^{1+\epsilon}}{d^2}\right\} & \text{if } \nu \ge 2 \end{cases}$$

*with high probability as $m$ gets large, where $2\sqrt{A} \le d \le A$ denotes the distance between the centers of the two clusters.*

*Proof of Proposition V.2.* First we consider the case where $\nu \ge 2$. The strategy for the proof is the following: in order to bound $\|H\|$, we divide the matrix into smaller blocks, apply the generalization of the classical Geršgorin discs' inequality presented in [8], and Lemma V.3 in order to bound the off-diagonal terms $\|H_{jk}\|$. For the diagonal terms $\|H_{jj}\|$, we reapply the generalized Geršgorin lemma and proceed in a recursive manner, until we reach small size blocks for which a loose estimate is sufficient to conclude.

Note that a network with area $A_0 = n^\nu$ has a density of $n^{1-\nu}$. This means that a cluster of area $A_1 = m_1 n^{\nu-1}$ contains $m_1$ nodes with high probability. Let us therefore decompose the network into $K_1$ square clusters of area $m_1 n^{\nu-1}$ with $m_1$ nodes each. Without loss of generality, we assume each cluster has exactly $m_1$ nodes and $K_1 = n/m_1 = A_0/A_1$. By the generalized Geršgorin lemma, we have

$$\|H\| \le \max\left\{\max_{1\le j\le K_1}\sum_{k=1}^{K_1}\|H_{jk}\|,\ \max_{1\le j\le K_1}\sum_{k=1}^{K_1}\|H_{kj}\|\right\} \quad (6)$$

where the $n \times n$ matrix $H$ is decomposed into blocks $H_{jk}$, $j, k = 1, \ldots, K_1$, with $H_{jk}$ denoting the $m_1 \times m_1$ channel matrix between cluster number $j$ and cluster number $k$ in the network. Let us also denote by $d_{jk}$ the corresponding inter-cluster distance, measured from the centers of these clusters. Based on Lemma V.3, we have

$$\|H_{jk}\|^2 \le \max\left\{\frac{m_1^{2+\epsilon}}{A_1 d_{jk}}, \frac{m_1^{1+\epsilon}}{d_{jk}^2}\right\} \overset{(a)}{=} \frac{m_1^{1+\epsilon}}{d_{jk}^2}$$

with high probability as $m_1 \to \infty$, where $(a)$ follows from the fact that $A_1/m_1 = n^{\nu-1} \ge n^{\nu/2} \ge d_j k$, since $\nu \ge 2$ (equivalently, $\frac{m_1}{A_1} \le \frac{1}{d_{jk}}$).

Let us now fix $j \in \{1, \ldots, K_1\}$ and define $R_j = \{1 \le k \le K_1 : d_{jk} < 2\sqrt{A_1}\}$ and $S_j = \{1 \le k \le K_1 : d_{jk} \ge 2\sqrt{A_1}\}$. By the above inequality, we obtain

$$\sum_{k=1}^{K_1}\|H_{jk}\| \le \sum_{k\in R_j}\|H_{jk}\| + \sqrt{n^\epsilon}\sum_{k\in S_j}\frac{\sqrt{m_1}}{d_{jk}}$$

with high probability as $m_1$ gets large. Observe that there are $8t$ clusters or less at distance $t\sqrt{A_1}$ from cluster $j$, so we obtain

$$\sum_{k\in S_j} \frac{\sqrt{m_1}}{d_{jk}} \le \sum_{t=2}^{\sqrt{K_1}} 8t\,\frac{\sqrt{m_1}}{t\sqrt{A_1}} = O\left(\sqrt{\frac{K_1 m_1}{A_1}}\right).$$

There remains to upper bound the sum over $R_j$. Observe that this sum contains at most 9 terms: namely the term $k = j$ and the 8 terms corresponding to the 8 neighboring clusters of cluster $j$. It should then be observed that for each $k \in R_j$, $\|H_{jk}\| \le \|H(R_j)\|$, where $H(R_j)$ is the $9m_1 \times 9m_1$ matrix made of the $9 \times 9$ blocks $H_{j_1,j_2}$ such that $j_1, j_2 \in R_j$. Finally, this leads to

$$\sum_{k=1}^{K_1} \|H_{jk}\| \le 9\|H(R_j)\| + 8\sqrt{n^\epsilon}\sqrt{\frac{K_1 m_1}{A_1}}$$

Using the symmetry of this bound and (6), we obtain

$$\|H\| \le 9 \max_{1\le j\le K_1} \|H(R_j)\| + 8\sqrt{n^\epsilon}\sqrt{\frac{K_1 m_1}{A_1}}. \qquad (7)$$

A key observation is now the following: For all $1 \le j \le K_1$, the $9M \times 9M$ matrix $H(R_j)$ has exactly the same structure as the original matrix $H$. Therefore, without loss of generality, let us assume $\|H_1\| = \max_{1\le j\le K_1} \|H(R_j)\|$. Finally, to bound $\|H_1\|$, the same technique may be reused recursively to get

$$\|H\| = O\left(\|H_l\| + \sqrt{n^\epsilon}\sum_{t=1}^{l}\sqrt{\frac{K_t m_t}{A_t}}\right)$$
$$= O\left(\|H_l\| + \sqrt{n^\epsilon}\sqrt{n^{1-\nu}}\sum_{t=1}^{l}\sqrt{K_t}\right),$$

where $m_i$ denotes the number of nodes in a square cluster of area $A_i$. Moreover, $K_i = A_{i-1}/A_i = m_{i-1}/m_i$ denotes the number of square clusters of area $A_i$ (and $m_i$ nodes) in a square network of area $A_{i-1}$ (and $m_{i-1}$ nodes). Note that $A_0 = A = n^\nu$ and $m_0 = n$. Finally, $\|H_i\|$ denotes the norm of the channel matrix of the network with square area $A_i$ and $m_i$ nodes.

Note that we have a trivial bound on $\|H_l\| = O\left(\sqrt{n^\epsilon}\, n^{1-\nu}\sqrt{A_l}\right)$ (see [10]) for any $\epsilon > 0$. Therefore, we have

$$\|H\| = O\left(\sqrt{n^\epsilon}\, n^{1-\nu}\sqrt{A_l} + \sqrt{n^\epsilon}\sqrt{n^{1-\nu}}\sum_{t=1}^{l}\sqrt{\frac{A_t}{A_{t-1}}}\right).$$

Upon optimizing over the $A_i$'s, we get $A_i = n^{\nu - \frac{i}{l+1}}$. As such, for $\nu \ge 2$, we get the desired result

$$\|H\| = O\left(n^{\frac{1-\nu}{2}}\, n^{\epsilon + \frac{1}{2\,(l+1)}}\right),$$

where for any $\epsilon' > \epsilon$, we can pick $l$ large enough so as $\epsilon' < \epsilon + \frac{1}{2\,(l+1)}$.

For $0 < \nu < 2$, we take the following approach: Notice that a dense network can be seen as a superposition of sparse networks. In other words, look at a network with $n$ nodes uniformly and independently distributed over an area $A = n^\nu$, as the superposition of $n^{1-\nu/2}$ networks, each of them having $m = n^{\nu/2}$ nodes uniformly and independently distributed over the same square area $A = n^\nu = m^2$. Again, by the generalized Geršgorin lemma, we obtain

$$\|H\| \le \sum_{k=1}^{n^{1-\nu/2}} \|H_{jk}\|$$

where the $n \times n$ matrix $H$ is decomposed into blocks $H_{jk}$, $j, k = 1, \dots, n^{1-\nu/2}$, with $H_{jk}$ denoting the $m \times m$ channel matrix between sparse network number $j$ and sparse network number $k$. Since each of these sparse networks has area $m^2$ with $m$ nodes, we can apply, $\forall\, j, k = 1, \dots, n^{1-\nu/2}$, the upper bound we got for $\nu = 2$ and obtain

$$\|H_{jk}\| = O\left(m^{-\frac{1}{2}+\epsilon}\right) = O\left(n^{-\frac{\nu}{4}+\frac{\epsilon}{2}}\right),$$

which results in

$$\|H\| = O\left(n^{1-\frac{3\nu}{4}+\frac{\epsilon}{2}}\right).$$

This finally proves Proposition V.2. □

## VI. Conclusion

In this work, we characterize the broadcast capacity of a wireless network at low SNR in line-of-sight environment and under various assumptions regarding the network density. The result exhibits a dichotomy between sparse networks, where node collaboration can hardly help enhancing communication rates, and constant density networks, where significant gains can be obtained via collaborative beamforming.

## VII. Acknowledgment

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