Proof of a conjecture by Gazeau et al. using the Gould Hopper polynomials

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We provide here some probabilistic interpretations of the generalized binomial distributions proposed by Gazeau et al.[1]. In the second part, we prove the "strong conjecture" expressed in[1] about the coefficients of the Taylor expansion of the exponential of a polynomial. The proof relies mainly on properties of the Gould-Hopper polynomials.

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I. TWO PROBABILISTIC APPROACHES

A. Introduction

A generalization of the classical binomial distributions is introduced by Gazeau et al.[1] as follows: assuming that $\{x_i\}$ is a sequence of positive numbers, the generalized factorial numbers are defined as

$$x_n! = \prod_{i=1}^{n} x_i$$

and the generalized binomial numbers as

$$\binom{x_n}{x_k} = \frac{x_n!}{x_k! x_{n-k}!}.$$

These coefficients are, in turn, used to define the sequence of polynomials $\{p_n\}_{n\geq 0}$ as the unique solution to the equation

$$\sum_{k=0}^{n} \binom{x_n}{x_k} \eta^k p_{n-k}(\eta) = 1, \ \forall n \in \mathbb{N}, \ \forall \eta \in \mathbb{R}. \tag{1}$$

The first values are

$$p_0(\eta) = 1, \ p_1(\eta) = 1 - \eta, \ p_2(\eta) = 1 - \frac{x_2}{x_1}\eta(1-\eta) - \eta^2.$$

Assuming that the polynomials $p_n(\eta)$ are positive, the values

$$\binom{x_n}{x_k} \eta^k p_{n-k}(\eta) \tag{2}$$

define a new probability distribution which is to be compared with the usual binomial distribution with parameter η, for which

$$\binom{n}{k} \eta^k (1-\eta)^{n-k}$$

is the probability of k successful events among n independent trials, each having a success probability η. Hence the new set of probabilities (2) extends the usual binomial setup, allowing to take into account correlation between these trials (see[1]).

The generating function

$$\mathcal{N}(t) = \sum_{n=0}^{+\infty} \frac{t^n}{x_n!}$$

and the generating function of the polynomials $p_n(\eta)$

$$G(t,\eta)=\sum_{n=0}^{+\infty}\frac{t^n}{x_n!}p_n(\eta)$$

are related as

$$G(t,\eta)=\frac{\mathcal{N}(t)}{\mathcal{N}(\eta t)}. \tag{3}$$

This is shown in[2] by expanding the product $\mathcal{N}(\eta t)\,G(t,\eta)$; however, this can also be proved by remarking that identity (1) is nothing but a convolution-type identity,

$$\left\{\frac{1}{x_n!}\right\}=\left\{\frac{\eta^n}{x_n!}\right\}*\left\{\frac{p_n(\eta)}{x_n!}\right\},$$

where the operator $*$ is the usual convolution operator that transforms two sequences $\{a_n\}$ and $\{b_n\}$ into the sequence

$$(a*b)_n=\sum_k a_k b_{n-k};$$

applying the convolution theorem yields $\mathcal{N}(\eta t)\,G(t,\eta)=\mathcal{N}(t)$.

In the two following subsections I B and I C, it is assumed that the coefficients x_n can be expressed as

$$x_n!=\frac{n!}{\mathbb{E}a^n} \tag{4}$$

for some random variable a with positive moments $\mathbb{E}a^n>0$. Here, $\mathbb{E}$ denotes the probabilistic expectation operator defined as

$$\mathbb{E}h(a)=\int h(a)dF(a)$$

where $F(a)$ is the probability distribution of the random variable a, and h is any function for which this integral is defined.

We deduce that the generating function

$$\mathcal{N}(t)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}\mathbb{E}a^n=\mathbb{E}\exp(at)=\phi_a(t)$$

coincides with the moment generating function ϕ_a of a.

B. Self-decomposability

A random variable a is self-decomposable if, for all $\eta\in[0,1]$, there exists a random variable a_η independent of a such that

$$a\sim\eta a+a_\eta \tag{5}$$

where $\sim$ denotes equality in distribution. Since, for a fixed value of η, the self-decomposability property implies that

$$\phi_a(t) = \phi_a(\eta t)\,\phi_{a_\eta}(t),$$

we deduce from (3) that the generating function of the polynomials $p_n(\eta)$ takes the form

$$G(t,\eta) = \phi_{a_\eta}(t).$$

Expanding each generating function on both sides, we deduce the following expression for the polynomials $p_n(\eta)$:

$$p_n(\eta) = \frac{\mathbb{E}a_\eta^n}{\mathbb{E}a^n}. \tag{6}$$

Note that by (5), the moment $\mathbb{E}a_\eta^n$ is a polynomial of maximum degree n in η : this can be shown by induction. For example, taking the expectation of both sides of (5) we deduce

$$\mathbb{E}a = \mathbb{E}\eta a + \mathbb{E}a_\eta$$

so that

$$\mathbb{E}a_\eta = (1-\eta)\,\mathbb{E}a$$

and by (6),

$$p_1(\eta) = 1-\eta.$$

As a conclusion, choosing any self-decomposable random variable with positive moments generates a sequence of factorial numbers $\{x_n!\}$ according to (4), from which the polynomials p_n can be computed according to (6).

As an example, let us consider a standard Gaussian random variable $N \sim \mathcal{N}(0,1)$ which satisfies

$$N \sim \eta N + N_\eta$$

with $N_\eta = \sqrt{1-\eta^2}N'$ where N' is again standard Gaussian and independent of N. Let us define the random variable

$$a = 1 + \sqrt{\alpha}N$$

so that a is self-decomposable with

$$a_\eta = 1 - \eta + \sqrt{\alpha\left(1-\eta^2\right)}N'$$

so that, using (6),

$$p_n(\eta) = \frac{\mathbb{E}\left(1-\eta+\sqrt{\alpha\left(1-\eta^2\right)}N'\right)^n}{\mathbb{E}\left(1+\sqrt{\alpha}N\right)^n}. \tag{7}$$

Since the Hermite polynomials read as moments[8] (p.49)

$$H_n(\eta) = 2^n \mathbb{E}\left(\eta + \imath\frac{N}{\sqrt{2}}\right)^n$$

we deduce

$$\frac{1}{x_n!} = \frac{\mathbb{E}\left(1+\sqrt{\alpha}N\right)^n}{n!} = \frac{\left(\frac{\alpha}{2}\right)^{\frac{n}{2}} H_n\left(\frac{\imath}{\sqrt{2\alpha}}\right)}{\imath^n n!}$$

and

$$p_n(\eta) = \left(1-\eta^2\right)^{\frac{n}{2}} \frac{H_n\left(\frac{\imath}{\sqrt{2\alpha}}\sqrt{\frac{1-\eta}{1+\eta}}\right)}{H_n\left(\frac{\imath}{\sqrt{2\alpha}}\right)}.$$

For example,

$$p_2(\eta) = (1-\eta)\left(1+\frac{\alpha-1}{\alpha+1}\eta\right).$$

C. Conjugated Random Variables

Another approach to the polynomials $p_n(\eta)$ is based on the notion of conjugated random variables, as introduced by Spitzer in[7]; the independent random variables a and b are conjugated if their moment generating functions φ_a and φ_b satisfy

$$\varphi_a(t)\,\varphi_b(t) = 1,\ \forall t \in \mathbb{R}.$$

We still assume that the coefficients x_n satisfy (4), but moreover, that the random variable a in (4) is conjugated to a random variable b. As a consequence

$$\frac{1}{\mathcal{N}(t)} = \frac{1}{\mathbb{E}e^{at}} = \mathbb{E}e^{bt}.$$

Defining as in[1] the coefficients I_n by

$$\frac{1}{\mathcal{N}(t)} = \sum_{n\ge 0}(-1)^n \frac{I_n}{x_n!}t^n,$$

we deduce

$$I_n = (-1)^n \frac{\mathbb{E}b^n}{\mathbb{E}a^n}.$$

Since the polynomials p_n satisfy[1]

$$p_n(\eta) = \sum_{k=0}^{n} (-1)^k \binom{x_n}{x_k} I_k \eta^k,$$

we deduce the expression

$$p_n(\eta) = \frac{\mathbb{E}(a+\eta b)^n}{\mathbb{E}a^n}.$$

As an example, it is well-known that N and $\imath N'$ are conjugated random variables if N and N' are independent standard Gaussian. As a consequence, $a = 1+\sqrt{\alpha}N$ and $b = -1+\imath\sqrt{\alpha}N'$ are also conjugated; the random variable

$$a + \eta b = (1-\eta) + \sqrt{\alpha}(N + \imath\eta N') \sim (1-\eta) + \sqrt{\alpha}\sqrt{1-\eta^2}N$$

which yields the polynomials p_n as in (7); hence in the Gaussian case, the self-decomposability and the conjugacy properties yield the same result.

D. The exponential of a Brownian motion

Another probabilistic approach to the coefficients x_n introduced in[1] is as follows: in[6], Carmona et al. consider the following random variable

$$I = \int_0^{+\infty} \exp(-\xi_s)\, ds \tag{8}$$

where the random variable ξ_s is called a subordinator with Laplace exponent $\Phi(t)$ defined by

$$\mathbb{E}e^{t\xi_s} = e^{s\Phi(t)}.$$

Carmona et al. proved the following result:

$$\mathbb{E}I^n = \frac{n!}{\Phi(1)\dots\Phi(n)},\ \ n \geq 1. \tag{9}$$

The elementary choice of a deterministic random process $\xi_s = s$ yields the usual binomial case

$$\Phi(n) = n;\ \ x_n = n!.$$

A physical interpretation of the representation

$$x_n = \Phi(n)$$

remains to be found.

II. PROOF OF TWO CONJECTURES BY GAZEAU ET AL.

In[1], the authors state the following conjecture, called *strong* conjecture:

Conjecture 1. *If* $\{a_i,\ 2 \le i \le p\}$ *are positive numbers, and with the notation* $x_n! = \prod_{k=1}^n x_k$*, then the numbers* x_i *such that*

$$\exp\left(t + \sum_{i=2}^{p} \frac{a_i}{i} t^i\right) = \sum_{n \ge 0} \frac{t^n}{x_n!}$$

satisfy the recurrence relation

$$x_{n+1} = \frac{n+1}{1 + \sum_{i=2}^{p} a_i \frac{x_n!}{x_{n-i+1}!}}.$$

In the following, we prove this conjecture using Gould-Hopper polynomials as defined in[3] and some integral representations of these polynomials as introduced in[4].

A. Preliminary Tools

In[5], Nieto and Truax consider the operator

$$I_j = \exp\left[\left(c\frac{d}{dx}\right)^j\right]$$

where c is a constant and j an integer. They remark that, for any well-behaved function f, I_1 acts as the translation operator

$$I_1 f(x) = f(x+c),$$

which can also be viewed as the probabilistic expectation

$$I_1 f(x) = \mathbb{E} f(x + cZ_1)$$

where Z_1 is the deterministic variable equal to 1.

In the case $j = 2$, with Z_2 denoting a Gaussian random variable with variance 2, I_2 acts as the Gauss-Weierstrass transform

$$I_2 f(x) = \mathbb{E} f(x + cZ_2).$$

It was shown in[5] that this result can be extended to any integer value of j as follows:

Proposition 1. *For any integer $j \geq 1$, there exists a complex-valued random variable Z_j such that the following representation*

$$I_j = \mathbb{E} f\left(x + cZ_j\right) \tag{10}$$

holds.

The properties of the complex-valued random variable Z_j were studied further in[4]. The only important property we need to know here is that

$$\mathbb{E}Z_j^k = \begin{cases} 0 & \text{if } k \neq 0 \mod j \\ \frac{(pj)!}{p!} & \text{if } k = pj,\ p \in \mathbb{N} \end{cases}.$$

and that, as a consequence, its characteristic function

$$\mathbb{E}\exp\left(uZ_j\right) = \exp\left(u^j\right),\ u \geq 0, \tag{11}$$

since a straightforward computation gives

$$\mathbb{E}\exp\left(uZ_j\right) = \sum_{k=0}^{+\infty} \frac{u^k}{k!} EZ_j^k = \sum_{p=0}^{+\infty} \frac{u^{pj}}{pj!}\frac{pj!}{p!} = \exp\left(u^j\right). \tag{12}$$

Definition 1. *The Gould-Hopper polynomials*[3] (p.58) *are defined as*

$$g_n^m\left(x, h\right) = \mathbb{E}\left(x + h^{\frac{1}{m}} Z_m\right)^n \tag{13}$$

and can be naturally generalized as

$$g_n\left(x, \mathbf{h}\right) = \mathbb{E}\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i\right)^n \tag{14}$$

for any vector $\mathbf{h} = [h_2, \ldots, h_p]$ such that $\{h_i \geq 0,\ 2 \leq i \leq p\}$.

Lemma 1. *The Gould-Hopper polynomials* (14) *satisfy the following identity*

$$g_n\left(x, \mathbf{h}\right) = \exp\left(\sum_{i=2}^{p} h_i \frac{d^i}{dx^i}\right) x^n. \tag{15}$$

Proof. We have

$$\exp\left(\sum_{i=2}^{p} h_i \frac{d^i}{dx^i}\right) x^n = \prod_{i=2}^{p} \exp\left(h_i \frac{d^i}{dx^i}\right) x^n$$

and by applying successively (10), we deduce the result. □

Lemma 2. *A generating function of the Gould-Hopper polynomials* $g_n(x,\mathbf{h})$ *is*

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} g_n(x,\mathbf{h}) = \exp\left(xt + \sum_{i=2}^{p} h_i t^i \right).$$

Proof. From the definition (13), we deduce, with $Z_2, \ldots, Z_p$ as in Proposition 1,

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{t^n}{n!} g_n(x,\mathbf{h}) &= \mathbb{E}\exp\left(t\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)\right) \\ &= \exp(xt) \prod_{i=2}^{p} \mathbb{E}\exp\left(t h_i^{\frac{1}{i}} Z_i \right)\end{aligned}$$

and the result follows from (12). □

From this lemma, we deduce that the factorial coefficients $x_n!$ satisfy

$$(x_n!)^{-1} = \frac{g_n(x,\mathbf{h})}{n!}.$$

In order to obtain a recurrence formula for the numbers x_n, we need the following recurrence relation on the Gould-Hopper polynomials.

Lemma 3. *The Gould-Hopper polynomials (14) satisfy the recurrence identity*

$$g_{n+1}(x,\mathbf{h}) = x g_n(x,\mathbf{h}) + \sum_{k=2}^{p} k h_k \frac{n!}{(n-k+1)!} g_{n+1-k}(x,\mathbf{h}).$$

Proof. The moment representation (14) yields

$$\begin{aligned}g_{n+1}(x,\mathbf{h}) = \mathbb{E}\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)^{n+1} &= \mathbb{E}\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)^{n} \\ = x\mathbb{E}\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)^{n} &+ \sum_{k=2}^{p} h_k^{\frac{1}{k}} \mathbb{E} Z_k \left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)^{n}.\end{aligned}$$

The first term is identified as $x g_n(x,\mathbf{h})$ and the second term is computed using the following lemma. □

Lemma 4. *The random variables* Z_j *as defined in Proposition 1 satisfy the Stein-type identity*

$$\mathbb{E}\left(Z_k f\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)\right) = k h_k^{1-\frac{1}{k}} \mathbb{E} f^{(k-1)}\left(x + \sum_{i=2}^{p} h_i^{\frac{1}{i}} Z_i \right)$$

for any smooth function f.

Proof. The partial derivative

$$\frac{\partial}{\partial h_k}\mathbb{E}\left(f\left(x+\sum_{i=2}^{p}h_i^{\frac{1}{i}}Z_i\right)\right)=\mathbb{E}\left(Z_kf'\left(x+\sum_{i=2}^{p}h_i^{\frac{1}{i}}Z_i\right)\right)\frac{1}{k}h_k^{\frac{1}{k}-1}$$

can also be computed from (15) as

$$\begin{aligned}\frac{\partial}{\partial h_k}\exp\left(\sum_{i=2}^{p}h_i\frac{d^i}{dx^i}\right)f(x)&=\exp\left(\sum_{i=2}^{p}h_i\frac{d^i}{dx^i}\right)\frac{d^k}{dx^k}f(x)\\&=\exp\left(\sum_{i=2}^{p}h_i\frac{d^i}{dx^i}\right)f^{(k)}(x)=\mathbb{E}f^{(k)}\left(x+\sum_{i=2}^{p}h_i^{\frac{1}{i}}Z_i\right)\end{aligned}$$

so that

$$\mathbb{E}\left(Z_kf'\left(x+\sum_{i=2}^{p}h_i^{\frac{1}{i}}Z_i\right)\right)\frac{1}{k}h_k^{\frac{1}{k}-1}=\mathbb{E}f^{(k)}\left(x+\sum_{i=2}^{p}h_i^{\frac{1}{i}}Z_i\right)$$

which is the result after replacing f' by f in both sides. Using this result with $f(x)=x^n$ yields the proof of Lemma 3. □

B. Proof of the conjecture

We can now prove the Conjecture 1 as follows: by Lemma 3, the quantities

$$(x_n!)^{-1}=\frac{1}{n!}g_n(x,\mathbf{h})$$

satisfy the recurrence

$$(n+1)(x_{n+1}!)^{-1}=x(x_n!)^{-1}+\sum_{k=2}^{p}kh_k(x_{n+1-k}!)^{-1}.$$

Dividing both sides by $(x_n!)^{-1}$, and remarking that

$$x_{n+1}^{-1}=\frac{(x_{n+1}!)^{-1}}{(x_n!)^{-1}},$$

we deduce

$$(n+1)x_{n+1}^{-1}=x+\sum_{k=2}^{p}kh_k\frac{x_n!}{x_{n+1-k}!}.$$

Choosing $h_k=\frac{a_k}{k}$ and $x=1$, we deduce the result.

We note that we have proved a result which is slightly more general than Conjecture 1, namely the fact that the coefficients x_n in the expression

$$\exp\left(xt+\sum_{i=2}^{p}\frac{a_i}{i}t^i\right)=\sum_{n\geq 0}\frac{t^n}{x_n!}$$

satisfy the recurrence

$$x_{n+1} = \frac{n+1}{x + \sum_{k=2}^{p} a_k \frac{x_n!}{x_{n+1-k}!}}.$$

Conjecture 1 in[1] corresponds to the case $x = 1$.

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