Let H be an $n \times n$ matrix whose entries are given by

$$H_{jk} = \frac{\exp(i\phi_{jk})}{\sqrt{j+k}}$$

where ϕ_{jk} are i.i.d. random phases uniformly distributed on $[0, 2\pi]$.
Let also $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ be the singular values of H.

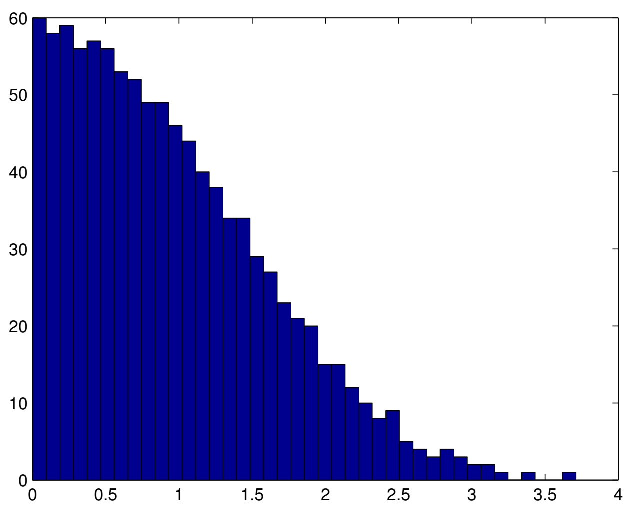
It can be shown, using the classical moments method, that

$$\mathbb{E}(\sigma_1) \leq 2\sqrt{\log n}$$

By an easy argument (namely, using the fact that $\sigma_1 \geq \sqrt{(HH^*)_{11}}$), it can also be shown that the order of this upper bound is correct, but the constant 2 seems to be loose, according to numerical simulations.

Open problem 0.1. *Can one say something more precise about the asymptotic behaviour of* $\mathbb{E}(\sigma_1)$ *or even* σ_1 *itself (almost sure result)?*

Moreover, the limiting distribution of the singular values seems to converge to a nice distribution in the large n limit, as the following figure shows.



Histogram of the singular values of H for $n = 1000$.

It can be shown that the Stieltjes transform of the asymptotic eigenvalue distribution of HH^* is given by $g(z) = \int_0^1 g(x,z)\,dx$, where $g(x,z)$ is the solution of the following integral equation:

$$g(x,z) = 1 \Bigg/ \left(-z + \int_0^1 \frac{1}{x+y}\, g(y,z)\,dy\right), \quad x \in [0,1],\ z \in \mathbb{C} : \operatorname{Im} z > 0$$

Open problem 0.2. *Can one characterize the solution of this integral equation and the corresponing eigenvalue distribution?*