Spatial Coupling and the Threshold Saturation Phenomenon

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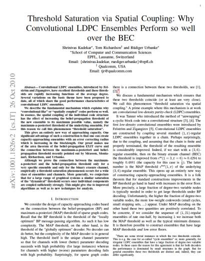
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

The latest version of these slides (Keynote and PDF) can be found at
https://ipg.epfl.ch/doku.php?id=en:publications:scc_tutorial

Part II: Three ways of characterizing ϵ^{Area}

Before we discuss how the threshold saturation phenomenon can be proved(,) let us discuss three alternative ways of how the area threshold can be characterized. For coding and transmission over the BEC these three characterizations are equivalent, but each adds some important facet. Further, depending on your background, some methods might seem more natural to you than others.

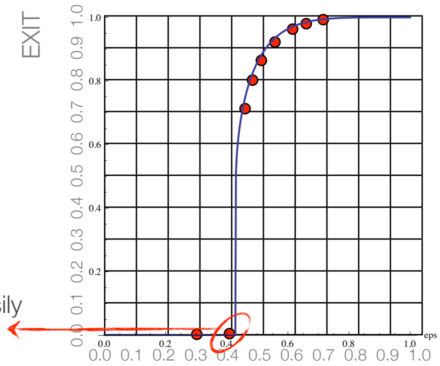

Maxwell Construction: The Hidden Bridge between Iterative and Maximum a Posteriori Decoding


Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well over the BEC

The Maxwell Characterization

One way to characterize the area threshold is by means of the Maxwell construction. This is the historically oldest approach and indeed the reason why the threshold is called the area threshold (since it is characterized by an area). This construction was introduced several years ago in the context of uncoupled ensembles with the aim of finding the MAP threshold of uncoupled codes. As it turns out, this threshold is also the BP threshold of coupled ensembles. Interestingly, originally when the Maxwell construction was introduced the the area threshold was defined, it was only possible to show that this threshold was an upper bound on the MAP threshold of uncoupled ensembles. But now, using the idea of coupling, it has recently been made possible to **prove** that the area threshold is equal to the MAP threshold. We start by explaining the construction and how it defines the area threshold. Only later on will we get back and explain how this construction relates to the MAP threshold.

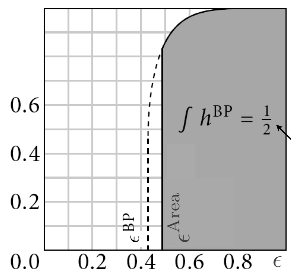
Recall - BP EXIT Curve



BP threshold easily characterized

Recall the BP EXIT curve from Part I and how it characterizes the BP threshold of the uncoupled ensemble: The BP threshold is the largest channel value for which the EXIT curve is equal to zero. For the (3,6) regular LDPC ensemble shown above, the BP threshold is ≈ 0.4299.

The Maxwell Construction and the Area Threshold



The area threshold is that channel parameter ϵ^{Area} so that the area enclosed under the EXIT curve from ϵ^{Area} to 1 is equal to the rate.

E.g., for the (3,6) code the rate is 1/2.

$\epsilon^{\mathrm{Area}} = 0.48818$

We define the area threshold, and denote it by ϵ^{Area}, as that channel value for which the area under the BP EXIT curve is equal to the design rate. For our running example of the (3,6) regular ensemble, the area threshold is equal to 0.48818, so it is considerably larger than the BP threshold. Also note that by construction the area threshold is always lower than the Shannon threshold since the EXIT curve is upper bounded by 1 and so the area threshold is by construction always below 1-rate.

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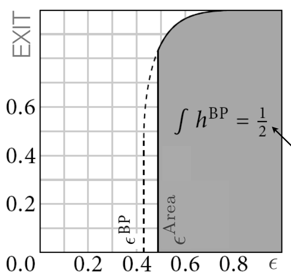

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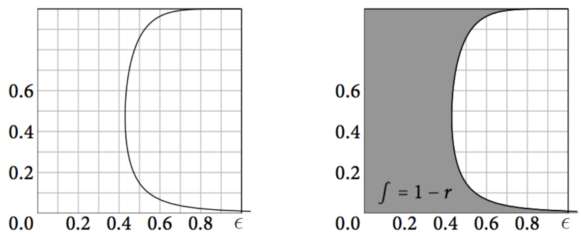
Why is this called the Maxwell construction?

At this point it is probably not clear why the Maxwell construction is called that. So let us explain the origin of the name in the next few slides.

A Simple Area Calculation



$$\text{EXIT}(x) = (1-(1-x)^{d_r-1})^{d_l} \qquad \epsilon(x) = \frac{x}{(1-(1-x)^{d_r-1})^{d_l-1}}$$

$$\int_0^1 \text{EXIT}(x)\epsilon'(x)\text{d}x = 1 - \frac{d_l}{d_r} = r$$

Let us provide another way of computing the area threshold from the BP EXIT curve. This method actually turns out to be useful in providing an operational interpretation of the area threshold (see "Maxwell Construction: The Hidden Bridge between Iterative and Maximum a Posterior Decoding", Measson, Montanari, Urbanke, 2005). We start with a simple calculation. Look at the EXIT curve. This curve looks like the 'C' shaped curve shown in slides. Let us integrate the area under this curve. A simple calculation shows that this area is equal to the rate of the code.

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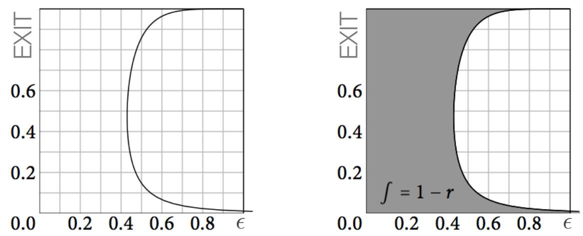

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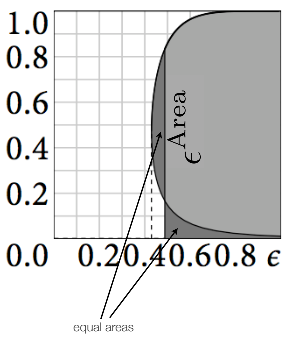

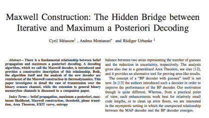

Combining this observation with the original definition of the area threshold it is not hard to see that at the area threshold the two areas shown in dark gray are equal. This means, an equivalent definition of the area threshold is to say that it is that point where a vertical line makes the two areas in dark gray to be of equal size. How does this interpretation explain the name? The reason for this name is that this construction is very similar in nature to the original Maxwell construction which was introduced by Maxwell to "correct" the equation of state of a gas proposed by van der Waals. This is also essentially what happens for coding. The EXIT curve is our equation of state and equivalent to the van der Waals equation. After we correct it we get the "actual equation of state" which for our case is the curve which characterizes the MAP decoder. For a more detailed explanation, please have a look at Chapter 15 in http://ipg.epfl.ch/lib/exe/fetch.php?media=en:courses:doctoral_courses_2012-2013:statphys.pdf

As we will see, this curve also characterizes the BP behavior of the coupled system.

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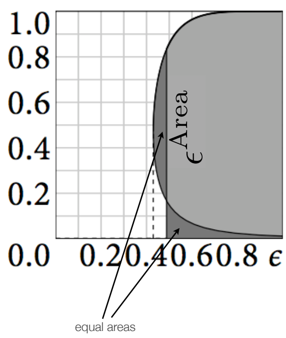

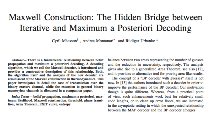

Ideal gas equation of state

$$pV = NkT$$

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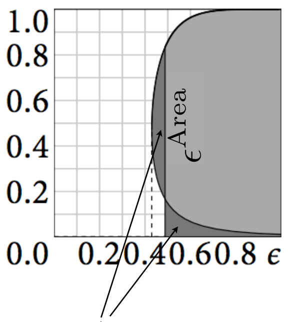

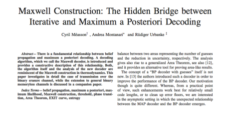

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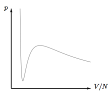
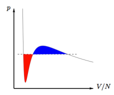
Original Maxwell construction applied to van der Waals equation of state:

$$(p + a\frac{N^2}{V^2})(V - bN) = NkT.$$



van der Waal



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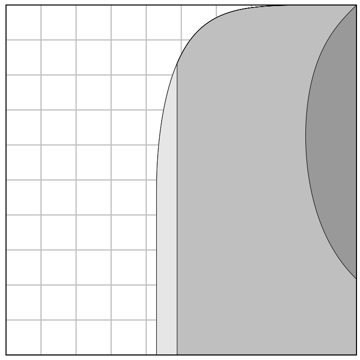
EXIT Curves for Coupled Ensembles and the Threshold Saturation Phenomenon



EXIT curves for increasing chain lengths L

$$\frac{1}{2L+1}\sum_{i=-L}^{L}(\frac{1}{w}\sum_{j=0}^{w-1} y_{i+j})^{d_l}$$

The sequence of plots above shows the EXIT curves for increasing chain lengths L. For very small L, the rate loss, is substantial and the effective rate is very small. Hence the EXIT curve is much further to the "right." But for larger and large L, the rate converges to the rate of the underlying ensemble (which in this case is the (3, 6) ensemble). Nevertheless we see that the EXIT curves do not converge to the EXIT curve of the underlying ensemble but follow the Maxwell construction.

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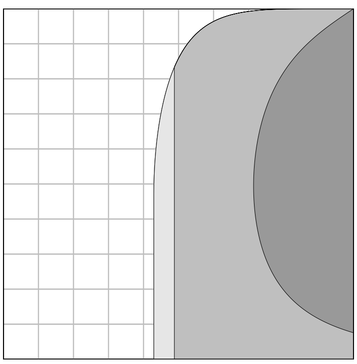

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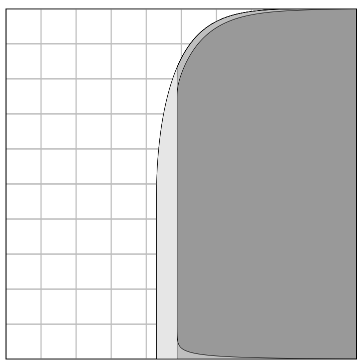

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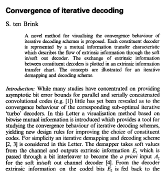

Convergence of iterative decoding

S. ten Brink

A novel method for visualising the convergence behaviour of iterative decoding schemes is proposed. Each constituent decoder is represented by a mutual information transfer characteristic which describes the flow of extrinsic information through the soft in/soft out decoder. The exchange of extrinsic information between constituent decoders is plotted in an extrinsic information transfer chart. The concepts are illustrated for an iterative demapping and decoding scheme.

Introduction: While many studies have concentrated on providing asymptotic bit error bounds for parallel and serially concatenated convolutional codes (e.g. [1]) little has yet been revealed as to the convergence behaviour of the corresponding sub-optimal iterative 'turbo' decoders. In this Letter a visualisation method based on bitwise mutual information is introduced which provides a tool for studying the convergence behaviour of iterative decoding schemes, yielding new design rules for improving the choice of constituent codes. For simplicity an iterative demapping and decoding scheme [2, 3] is considered in this Letter. The demapper takes soft values from the channel and outputs extrinsic information E_1 which is passed through a bit interleaver to become the *a priori* input A_2 for the soft in/soft out channel decoder [4]. From the decoder extrinsic information on the coded bits E_2 is fed back to the

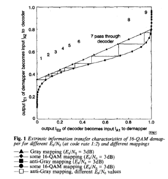

Fig. 1 *Extrinsic information transfer characteristics of 16-QAM demapper for different E_b/N_0 (at code rate 1:2) and different mappings*

- Gray mapping (E_b/N_0 = 3dB)
- some 16-QAM mapping (E_b/N_0 = 3dB)
- anti-Gray mapping (E_b/N_0 = 3dB)
- some 16-QAM mapping (E_b/N_0 = 3dB)
- anti-Gray mapping, different E_b/N_0 values

Wave-Like Solutions of General One-Dimensional Spatially Coupled Systems

arXiv:1208.5273v1 [cs.IT] 27 Aug 2012

I. Introduction

The EXIT Chart Characterization

The second characterization is in terms of EXIT charts. EXIT charts were introduced by S. ten Brink as a convenient way of visualizing DE. For transmission over the BEC the EXIT chart method is equivalent to DE and so it is exact. For general channels it is no longer exact but still gives a nice and important engineering insight into the problem and typically the predicted thresholds are good approximations. *A small word of warning:* **EXIT charts** and **EXIT curves** which we introduced previously are quite different objects despite their similar name. So it is important not to confuse the two. For the EXIT charts visualize the actions of the two operations of the iterative decoder, whereas the EXIT curve represents the behavior of the overall code. The reason both objects have the word "EXIT" in there is that in both cases we measure the same thing, but once we make local measurements (EXIT charts), whereas in the other case we measure the global behavior (EXIT curve).

DE and EXIT Charts

$$x^{(\ell)} = \epsilon(y^{(\ell)})^{d_l-1} \quad y^{(\ell)} = 1-(1-x^{(\ell-1)})^{d_r-1}$$

The idea of EXIT functions is that we plot the two component functions in a square in such a way that the points of DE appear like a "staircase" bound by these two component functions. In this way we immediately see the following. The DE points converge to zero if and only if the two curve do not cross. Further, it is not hard to see that we have a "good" system if they two curves are as closely matched as possible. Indeed, if we could match them over the whole range we would have designed a capacity-achieving code. This is called the "matching" condition.

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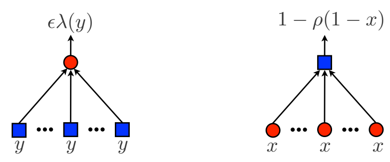

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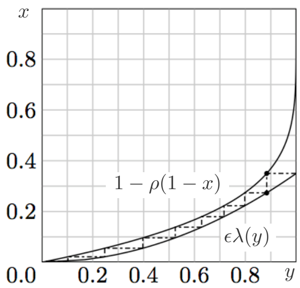

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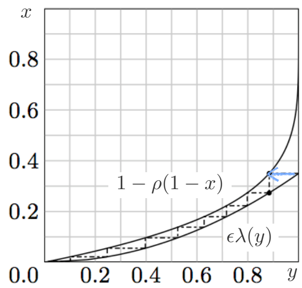

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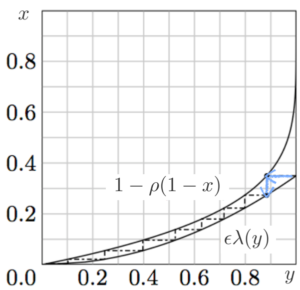

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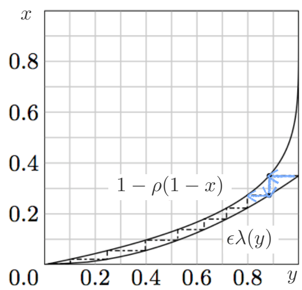

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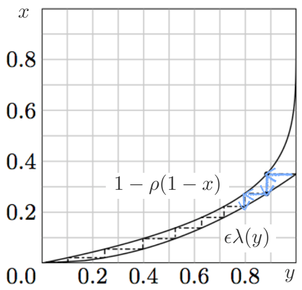

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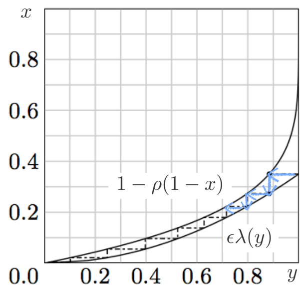

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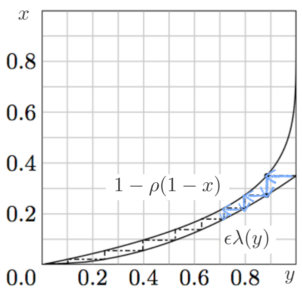

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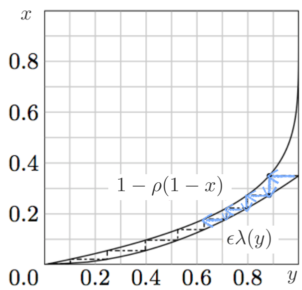

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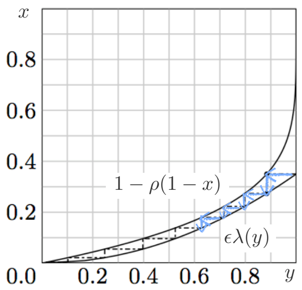

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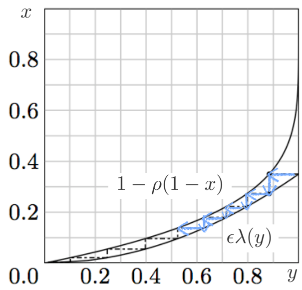

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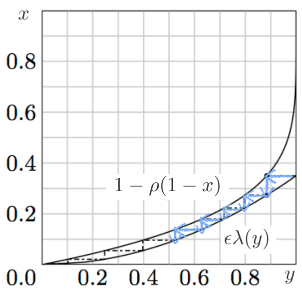

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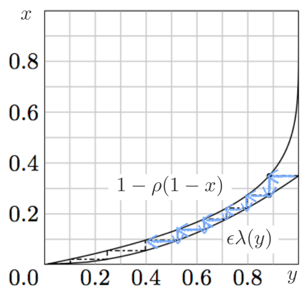

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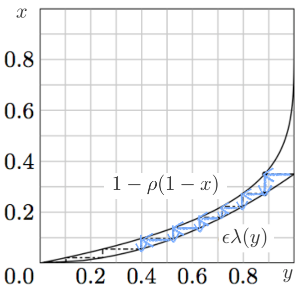

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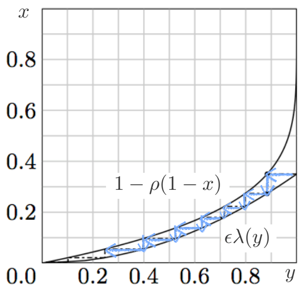

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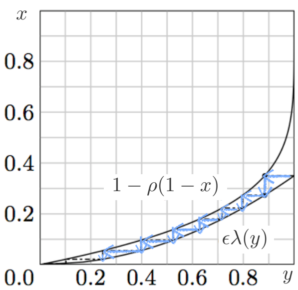

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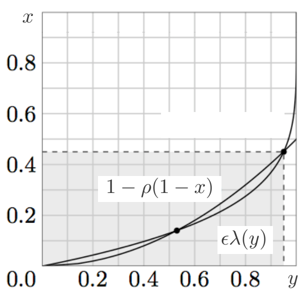

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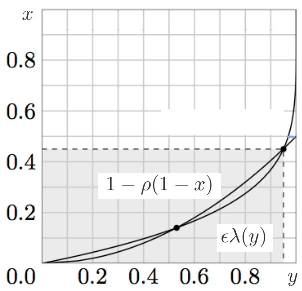

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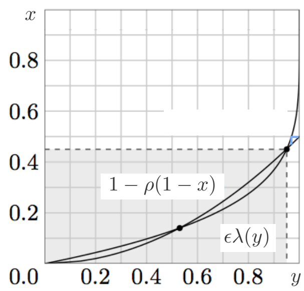

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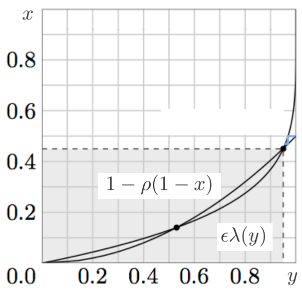

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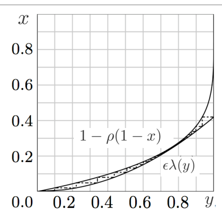
Thresholds -- Uncoupled versus Coupled



BP threshold for uncoupled:
Matching of *curves*

If you like EXIT functions then the following is probably the nicest pictorial description of how to determine the threshold under spatial coupling. If we look at the uncoupled case then we know that the threshold is given by that channel parameter so that the two EXIT curves describing the action at the check and variable nodes just touch but do not cross. If we go to coupled systems this criterion is relaxed. The two EXIT curves are now allowed to cross but not by too much. Indeed, the threshold is given a balance of areas. **Note that one can show that this condition for the threshold is EXACTLY the same as the matching of areas condition which we had for the Maxwell construction.** So this is not a new condition. It is the same condition but represented graphically in a different way.

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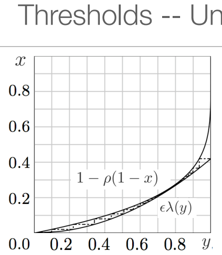

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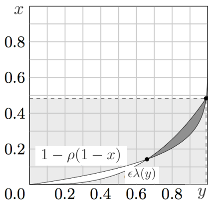

BP threshold for coupled:
Matching of *areas*

white area must be at least as large as dark gray area; threshold when in balance
NOTE: exactly the same condition as before

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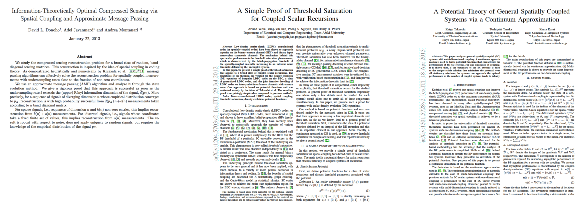
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A Potential Theory of General Spatially-Coupled Systems via a Continuum Approximation

The Potential Function Characterization

Let us now discuss the third way of characterizing the threshold for spatially coupled systems. Again, it gives exactly the same condition, even though this might not be completely obvious at first sight. Several groups of authors have suggested a potential function approach to the problem. Our notation will follow the lead of Yedla, Jian, Nguyen, and Pfister.

Potential Function

$$U(x, \epsilon) = \int_0^x (z - \epsilon\lambda(1-\rho(1-z)))\rho'(1-z)\mathrm{d}z$$

Asking that the potential function is increasing for all x>=0 (and the chosen value of eps) is equivalent to asking that the DE recursions will converge to 0. So the characterization of the BP threshold in terms of the potential function is quite straightforward.

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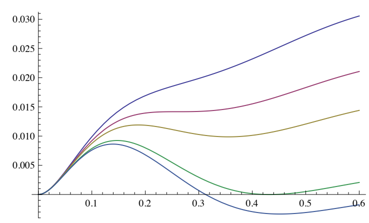

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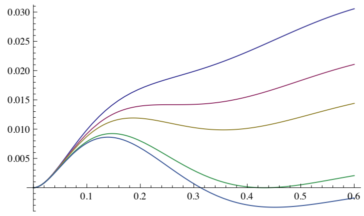

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iff non-negative for all z then BP converges

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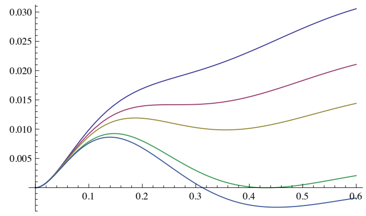

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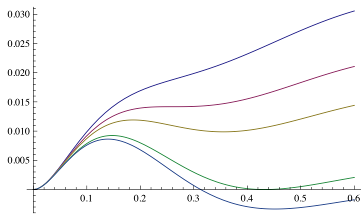

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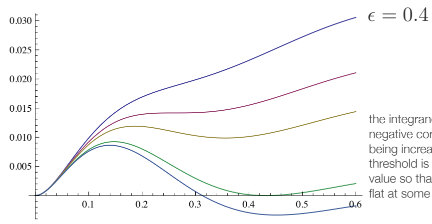

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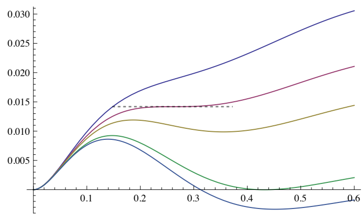

$\epsilon = 0.4$

$\epsilon^{\mathrm{BP}} = 0.42944$

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Asking that the potential function is increasing for all x>=0 (and the chosen value of eps) is equivalent to asking that the DE recursions will converge to 0. So the characterization of the BP threshold in terms of the potential function is quite straightforward.

Potential Function

$$\begin{aligned} U(x,\epsilon) = & \int_0^x (z - \epsilon\lambda(1-\rho(1-z)))\rho'(1-z)\mathrm{d}z \\ = & \; x(1-\rho(1-x)) - R(1-x) - \epsilon\Lambda(1-\rho(1-x)) \end{aligned}$$

For the area threshold we are picking the largest eps value so that the potential function is positive for all x>=0. The picture above explains why this is exactly the same as the balance of area condition which we previously saw for the EXIT functions. Let us go over the argument in detail. We want U(x, eps) to be non-negative. Clearly we only have to check this condition at all extreme points of U(x, eps), i.e., at all those points x so that U'(x, eps) is equal to zero. But at these points we see from the definition of the potential function that x and eps are related by the DE equation, i.e., x is a FP of DE for parameter eps. This means that in the EXIT picture below the gray box indicates exactly the FP of DE and for this corner point of this gray box x and y are related by the DE equations. We can now interpret the three terms of the explicit evaluation of U above as areas in this figure and the condition that U is non-negative implies that the left-most of the two enclosed areas must be no smaller than the right-most one. This is exactly the condition we saw previously.

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for all x >= 0

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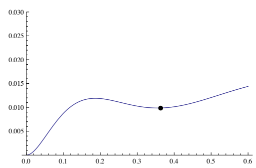

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Claim: this is exactly the same condition as requiring that the two areas are in balance as we have seen this for the EXIT curves

Note: Asking that U(x, eps)>=0 (where eps is fixed) is the same as asking that the minimum of U(x, eps) is above 0. But at the minimum we have U'(x, eps)=0 and so eps and x form FP of DE.

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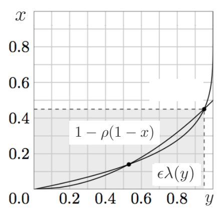
IF x and eps form a FP of DE:



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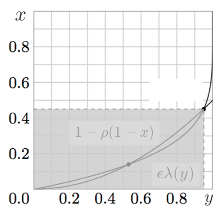

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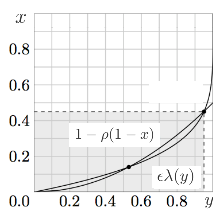
IF x and eps form a FP of DE: gray area

area to the left of $1-\rho(1-x)$ **in the gray box**



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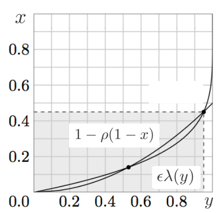
IF x and eps form a FP of DE: gray area area to the left of $1-\rho(1-x)$ in the gray box

area under $\epsilon\lambda(y)$ **in the gray box**



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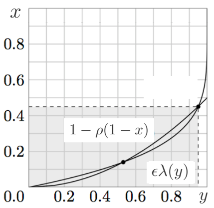
IF x and eps form a FP of DE: gray area | area to the left of $1-\rho(1-x)$ in the gray box | area under $\epsilon\lambda(y)$ in the gray box



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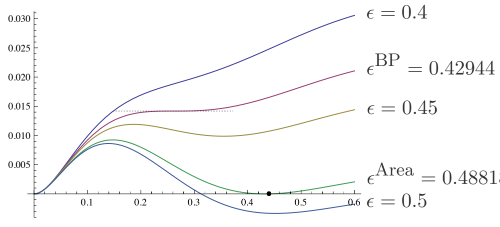
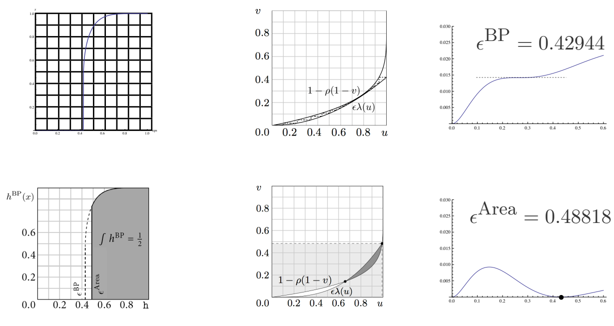
Potential Function



Think of the height of the curve as an ``energy."
The BP solution is at the first local minimum from the right.
The globally best solution is at the global minimum.
The correct solution is at zero.

We can think of the height of the potential function as an "energy" which the system can take on. The system starts on the right and tries to minimize its energy. It gets stuck in the first local minimum counted from the right. So if the curve is not increasing then we get stuck in the wrong point. This determines the BP threshold. If the curve ever dips below zero then the global minimum is no longer the one at 0. This determines the area threshold.

Why three approaches?



Historically, it is the first criterion. Also, it looks exactly the same when we extend to general BMS channels. Nice connections to the original Maxwell construction and the starting point to determine the MAP threshold of a code.

EXIT charts are widely used in the coding and communications literature already. Exact for the BEC and often good approximation for general cases.

Leads to the currently simplest known proof of the threshold saturation phenomenon for 1D coupled systems. Can be extended to general systems, but of course no longer has this nice and simple graphical interpretation.

Why do we show three different criteria here? For the case we consider, namely coding and transmission over the BEC, the three criteria are all equivalent and one would suffice. The reason for nevertheless showing all three is that they each have their advantage. **The Maxwell construction** has nice connections to problems in statistical physics (where the original Maxwell construction comes from) and the picture is exactly the same (although the proofs are more complicated) if we consider transmission over general BMS channels. Further, it strongly suggests that the area threshold is also the MAP threshold of the underlying ensemble. Indeed, that this is true has recently been shown by Giurgiu, Macris and Urbanke, http://arxiv.org/pdf/1301.5676.pdf. **The EXIT chart approach** might be the most familiar. In particular, EXIT charts and the matching condition are frequently used to analyze systems exactly which have a one-dimensional state or to approximately model more general systems (e.g., Gaussian approximation). So for any such system, if we replace the matching condition with the area balance condition then we get the equivalent criterion for coupled systems. If the original state is one dimensional then this criterion is exact, otherwise it is an approximation in the same way as the matching condition for EXIT charts is an approximation for uncoupled systems. Finally, the **potential function approach** leads to the currently simplest known proof for one dimensional systems. It can be extended to systems whose

The connection between the area and the MAP threshold.

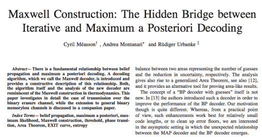
Originally the area threshold was not introduced to study spatially coupled codes but to determine the MAP threshold of uncoupled codes. Let us quickly explain this connection in some more detail.

Maxwell Construction and MAP Threshold

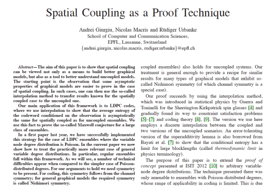
For the BEC it was shown that

$$\epsilon^{\mathrm{MAP}} = \epsilon^{\mathrm{Area}}$$

and the area threshold is an upper bound on the MAP threshold for all BMS channels.


Maxwell Construction: The Hidden Bridge between Iterative and Maximum a Posteriori Decoding

Cyril Méasson, Andrea Montanari and Rüdiger Urbanke


Spatial Coupling as a Proof Technique

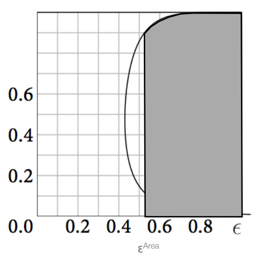
Andrei Giurgiu, Nicolas Macris and Rüdiger Urbanke

School of Computer and Communication Sciences, EPFL, Lausanne, Switzerland

It was also shown in “Maxwell Construction: The Hidden Bridge between Iterative and Maximum a Posterior Decoding”, Measson, Montanari, Urbanke, 2005, that ϵ^{MAP} is equal to ϵ^{Area} for the BEC and that the area threshold is always an upper bound on the MAP threshold for the general case. One can find a proof (for the BEC case) also in the book Modern Coding Theory, Theorem 3.121 on page 126. Thus one can obtain the ϵ^{MAP} from the balancing of areas as shown in the slides. This gives a very simple tool for computing a seemingly difficult quantity, the MAP threshold. It was conjectured that this is also true for general BMS channels. In a recent work “Spatial Coupling as a Proof Technique”, Giurgiu, Macris, Urbanke, http://arxiv.org/pdf/1301.5676.pdf, it was shown that this is indeed true and the proof technique involved using spatially coupled codes!

Proof Outline

$$\begin{aligned}
& r - \frac{1}{n}H(X \mid Y(\epsilon^{\mathrm{Area}})) \\
=& \frac{1}{n}H(X \mid Y(1)) - \frac{1}{n}H(X \mid Y(\epsilon^{\mathrm{Area}})) \\
=& \int_{\epsilon^{\mathrm{Area}}}^{1} \frac{\mathrm{d}H(X \mid Y(\epsilon))}{n\mathrm{d}\epsilon}\mathrm{d}\epsilon \\
=& \int_{\epsilon^{\mathrm{Area}}}^{1} \sum_{i=1}^{n} \frac{\partial H(X \mid Y(\epsilon))}{n\partial\epsilon_i}\mathrm{d}\epsilon \\
=& \int_{\epsilon^{\mathrm{Area}}}^{1} \sum_{i=1}^{n} \frac{\partial[H(X_i \mid Y(\epsilon)) + H(X_{\sim i} \mid X_i, Y_{\sim i})]}{\partial\epsilon_i}\mathrm{d}\epsilon \\
=& \int_{\epsilon^{\mathrm{Area}}}^{1} \sum_{i=1}^{n} \frac{\partial H(X_i \mid Y(\epsilon))}{n\partial\epsilon_i}\mathrm{d}\epsilon \\
=& \int_{\epsilon^{\mathrm{Area}}}^{1} \sum_{i=1}^{n} \frac{\partial H(X_i \mid Y_i(\epsilon_i), Y_{\sim i}(\epsilon))}{n\partial\epsilon_i}\mathrm{d}\epsilon \\
=& \int_{\epsilon^{\mathrm{Area}}}^{1} \frac{1}{n}\sum_{i=1}^{n} H(X_i \mid Y_{\sim i}(\epsilon))\mathrm{d}\epsilon \\
\leq& \int_{\epsilon^{\mathrm{Area}}}^{1} \mathrm{EXIT}(\epsilon)\mathrm{d}\epsilon \quad n \to \infty \\
=& r
\end{aligned}$$



Let us quickly show how to prove that the area threshold is an upper bound on the MAP threshold. We focus on the BEC to keep things simple.
As the sequence of inequalities shows, when n tends to infinity, then the integral under the EXIT curve is an upper bound on the 1-r-1/n H(X | Y(ϵ^{Area})). If we cancel r from both sides we see that 1/n H(X | Y(ϵ^{Area})) is strictly positive above the area threshold. So if the conditional entropy is positive for this channel parameter, then no decoder can hope to find the transmitted codeword reliably, not even then MAP decoder. In other words, ϵ^{Area} is an upper bound on ϵ^{MAP}.

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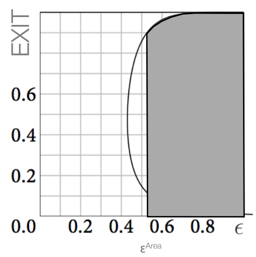

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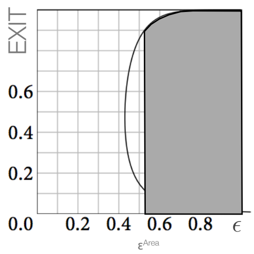
\end{aligned}
$$



We conclude that H(X | Y(ε^Area)) >=0, which means that we transmit above the MAP threshold.

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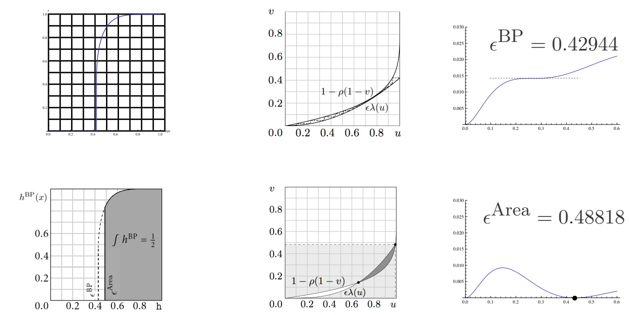

Historically, it is the first criterion. Also, it looks exactly the same when we extend to general BMS channels. Nice connections to the original Maxwell construction and the starting point to determine the MAP threshold of a code.

EXIT charts are widely used in the coding and communications literature already. Exact for the BEC and often good approximation for general cases.

Leads to the currently simplest known proof of the threshold saturation phenomenon for 1D coupled systems. Can be extended to general systems, but of course no longer has this nice and simple graphical interpretation.

Why do we show three different criteria here? For the case we consider, namely coding and transmission over the BEC, the three criteria are all equivalent and one would suffice. The reason for nevertheless showing all three is that they each have their advantage. **The Maxwell construction** has nice connections to problems in statistical physics (where the original Maxwell construction comes from) and the picture is exactly the same (although the proofs are more complicated) if we consider transmission over general BMS channels. Further, it strongly suggests that the area threshold is also the MAP threshold of the underlying ensemble. Indeed, that this is true has recently been shown by Giurgiu, Macris and Urbanke, http://arxiv.org/pdf/1301.5676.pdf. **The EXIT chart approach** might be the most familiar. In particular, EXIT charts and the matching condition are frequently used to analyze systems exactly which have a one-dimensional state or to approximately model more general systems (e.g., Gaussian approximation). So for any such system, if we replace the matching condition with the area balance condition then we get the equivalent criterion for coupled systems. If the original state is one dimensional then this criterion is exact, otherwise it is an approximation in the same way as the matching condition for EXIT charts is an approximation for uncoupled systems. Finally, the **potential function approach** leads to the currently simplest known proof for one dimensional systems. It can be extended to systems whose