Discrete Structures | EPFL, Fall 2013

Midterm Exam – Solutions

Date: 12.11.2013 | **Version A**

MULTIPLE-CHOICE QUESTIONS

1. B.

2. C.

3. B.

4. B.

5. A.

6. i) F ii) F iii) F iv) T

7. i) B ii) C iii) B iv) B v) A vi) A

8. D

9. i) B ii) C iii) A iv) B

10. A

11. C

12. C–A–D–B–E

13. B

PROBLEMS

14.

1. The statement is true. The proof follows.

   Since $f_1 = \Theta(f_2)$, there exists x_0 and constants $c_1 > 0$ and c_2 s.t. for all $x \geq x_0$,

   $$c_1|f_2(x)| \leq |f_1(x)| \leq c_2|f_2(x)|.$$

   Since $f_1(x) > 0$, then $c_2 > 0$. Indeed, if $c_2 \leq 0$, the inequality above cannot be satisfied. As the function $h(x) = x^{-13}$ is decreasing for all $x > 0$, we obtain that

   $$h(c_1|f_2(x)|) \geq h(|f_1(x)|) \geq h(c_2|f_2(x)|),$$

   which implies that

   $$(c_2)^{-13}(|f_2(x)|)^{-13} \leq (|f_1(x)|)^{-13} \leq (c_1)^{-13}(|f_2(x)|)^{-13}.$$

   Consequently, there exists x_0' and constants $c_1' > 0$ and c_2' s.t. for all $x \geq x_0'$,

   $$c_1'(|f_2(x)|)^{-13} \leq (|f_1(x)|)^{-13} \leq c_2'(|f_2(x)|)^{-13}.$$

   Indeed, it is enough to take $x_0' = x_0$, $c_1' = (c_2)^{-13} > 0$, and $c_2' = (c_1)^{-13}$.

2. The statement is false. Indeed, pick $f_1(x) = x$ and $f_2(x) = 2x$. Then, clearly $f_1(x) = \Theta(f_2)$. By definition $g_1(x) = 11^x$, and $g_2(x) = 11^{2x} = 121^x$. Therefore, it is not true that $g_1 = \Omega(g_2)$.

15.

Base step. If $n = 0$, the left hand side and the right hand side of the equality are both 0.

Induction step. Assume that $\displaystyle\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Then,

$$\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$
$$= \frac{(n+1)^2(n^2+4(n+1))}{4} = \frac{(n+1)(n+2)^2}{4}.$$