Solution to Graded Problem Set 2

Date: 27.09.2013 Due date: 4.10.2013

Problem 1.

a) Truth table shows that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

b) Apply rules successively:

$$
\begin{aligned}
& (p \to q) \wedge (\neg p \to (\neg q \to (p \wedge \neg p))) & \\
\equiv\ & (p \to q) \wedge (\neg p \to (\neg q \to \mathrm{F})) & \text{negation law} \\
\equiv\ & (\neg p \vee q) \wedge (\neg\neg p \vee (\neg\neg q \vee \mathrm{F})) & \text{definition of implication} \\
\equiv\ & (\neg p \vee q) \wedge (p \vee (q \vee \mathrm{F})) & \text{double negation law} \\
\equiv\ & (\neg p \vee q) \wedge (p \vee q) & \text{identity law} \\
\equiv\ & (\neg p \wedge p) \vee q & \text{distributivity law} \\
\equiv\ & \mathrm{F} \vee q & \text{negation law} \\
\equiv\ & q & \text{identity law}
\end{aligned}
$$

Problem 2.

1. **True.** Indeed, $P(1,3)$, $P(2,1)$, and $P(3,1)$ are true.
2. **True.** Indeed, $P(2,1)$, $P(2,2)$, and $P(2,1)$ are true.
3. **False.** Indeed, $P(1,3)$, and $P(3,1)$ are true.
4. **True.** Indeed, it is enough to pick $x = y$ and to remind that $F \to F$ and $T \to T$.
5. **True.** Indeed, $P(1,3)$, $P(2,1)$, and $P(2,3)$ are true.
6. **False.** Indeed, $P(1,1)$ is false.

Problem 3.

1. $\neg\forall A F(A)$ or $\exists A \neg F(A)$.

2. $\forall A \forall B[(S(A,B) \wedge F(B)) \to F(A)]$.

3. $\neg \exists A \exists B[(\neg F(A)) \wedge S(A,B) \wedge F(B)]$.

Problem 4. Let $R(x)$ be the predicate "x has read the textbook" and $P(x)$ be the predicate "x passed the exam". The following is the proof :

1. $\forall x(R(x) \to P(x))$	hypothesis
2. $R(\text{Ed}) \to P(\text{Ed})$	universal instantiation on (1)
3. $R(\text{Ed})$	hypothesis
4. $P(\text{Ed})$	modus ponens on (2) and (3)

Problem 5. We define the following logical functions : $P(x)$ as "x has a bachelor degree in Mathematics", $R(x)$ as "x has more than 125 credits", $S(x)$ as "x passed the exam *Analysis II*", and $T(x)$ as "x passed the exam *Analysis I*".

1. $\exists x(P(x) \wedge \neg R(x))$	premise
2. $P(a) \wedge \neg R(a)$	existential instantiation from (1)
3. $P(a)$	simplification from (2)
4. $\forall x(P(x) \to S(x))$	premise
5. $P(a) \to S(a)$	universal instantiation from (4)
6. $S(a)$	modus ponens from (3) and (5)
7. $\neg R(a)$	simplification from (2)
8. $\forall x(T(x) \to R(x))$	premise
9. $T(a) \to R(a)$	universal instantiation from (8)
10. $\neg T(a)$	modus tollens from (7) and (9)
11. $S(a) \wedge \neg T(a)$	conjunction from (6) and (10)
12. $\exists x(S(x) \wedge \neg T(x))$	existential generalization from (11)

Problem 6. Let p = "you tell false" and q = "ruins on the left". The question to be answered is $p \oplus q$, i.e., $\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$. If the woman of Questionland says yes (=true), the ruins are on the left, otherwise they are on the right, regardless of the fact that he says the truth or not.