Midterm Exam - Solutions

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Problem 1.

(i) Let us prove that $\mathcal{M}_X$ is a matroid.

Clearly, the empty set $\emptyset$ is in $\mathcal{I}_X$, because it induces the degree 0 on all the elements of X.

By removing edges from $A \in \mathcal{I}_X$, the induced degrees on X cannot increase and, hence, they remain either 0 or 1. Therefore, for any $B \subseteq A$, $B \in \mathcal{I}_X$.

As concerns the third property, observe that, for any $A \in \mathcal{I}_X$, the number of vertices of X with induced degree equal to 1 is $|A|$. Hence, if $|A| < |B|$, there exists a vertex $\bar{v}$ which is touched by an edge $\bar{e} \in B \setminus A$. As a result, $A \cup \{\bar{e}\} \in \mathcal{I}_X$.

An analogous proof holds for $\mathcal{M}_Y$.

(ii) By definition, M is a matching if and only if the induced maximum degree both on X and on Y is 1. This occurs if and only if M is both in $\mathcal{I}_X$ and in $\mathcal{I}_Y$.

Problem 2.

(i) Take an MST and let $\bar{e}$ be an edge of maximum weight. In a tree, any edge is a cut edge and, hence, by deleting $\bar{e}$ we define a partition of the vertices (X, Y) s.t. $V = X \cup Y$. Any spanning tree of G can be generated in the following way: find a spanning tree for the subgraph with vertex set X and a spanning tree for the subgraph with vertex set Y; then, add to these two spanning trees any edge in E which connects X to Y. Clearly, in order to get an MST we must pick from all those edges an edge of minimum cost. Since, by assumption, we started with an MST there cannot exist an edge of a spanning tree with smaller cost than $w(\bar{e})$.

(ii) Take two cycles and join them by one edge. Let the joining edge have weight 3 and in each cycle let all the edges have weight 1 except one edge which has weight 2. In order to obtain an MBST, it suffices to delete any edge from the left cycle and any edge from the right cycle. However, only if we delete the two edges of weight 2, we obtain an MST.

Problem 3. Consider the following bipartition of V, $V = X \cup Y$, where X consists of all subsets of $\{1, 2, \cdots, n\}$ of even cardinality and Y consists of all such subsets of odd cardinality. Clearly the union of these two sets equals V and the union is disjoint.

Consider two subsets of $\{1, 2, \cdots, n\}$, call them A and B. Assume without loss of generality that $|A| \leq |B|$. Then, in order for the symmetric difference of A and B to have cardinality 1, we must have $A \subset B$ and B must contain exactly 1 more element than A, i.e., $|A| = |B| + 1$. Hence if A is in X then B must be in Y and vice versa.

Problem 4. Let $a_{i,j}^{(n)}$ be the number of walks of length n from vertex i to vertex j for $i, j \in \{1, 2, 3\}$. Then, as we have seen in Problem Set 2, $a_{i,j}^{(n)}$ is the (i, j)-th entry of A^n. Our goal is to estimate $a_{1,1}^{(n)}$.

Now, by definition of eigenvalue and eigenvector,

$$A^n x = \lambda^n x.$$

Hence,

$$a_{1,1}^{(n)} + \frac{a_{1,2}^{(n)}}{\sqrt{2}} + \frac{a_{1,3}^{(n)}}{\sqrt{2}} = \lambda^n,$$

which, as the elements of A^n are all positive, gives us the upper bound

$$a_{1,1}^{(n)} \leq \lambda^n.$$

Similarly we have that

$$a_{1,1}^{(n-1)} + \frac{a_{1,2}^{(n-1)}}{\sqrt{2}} + \frac{a_{1,3}^{(n-1)}}{\sqrt{2}} = \lambda^{n-1}.$$

This implies that

$$a_{1,1}^{(n-1)} + a_{1,2}^{(n-1)} + a_{1,3}^{(n-1)} \geq \lambda^{n-1}.$$

Observe that there is an edge from vertex 1 to any other vertex (included vertex 1 itself), which means that any walk of length n from vertex 1 back to vertex 1 is the union of a walk of length $n-1$ from vertex 1 to vertex i ($i \in \{1, 2, 3\}$) and the edge from vertex i to vertex 1. Therefore $a_{1,1}^{(n)} = a_{1,1}^{(n-1)} + a_{1,2}^{(n-1)} + a_{1,3}^{(n-1)}$, which gives the lower bound

$$a_{1,1}^{(n)} \geq \lambda^{n-1}.$$

The bounds are good since their ratio is $\lambda = 1 + \sqrt{2}$ and it stays constant as n goes large.