## 3) First set of rules of Quantum mechanics.

Here we infer the first set of rules we have "arrived at" in the previous paragraph. In this chapter we keep the discussion a bit informal and will revisit this subject in next chapter.

a) A quantum system (isolated from the rest of the world) can be described by a state vector $|\psi\rangle \in \mathcal{H}$ living in a Hilbert space $\mathcal{H}$. A Hilbert space is vector space endowed with a scalar product (for finite dimensional spaces this is ok, for inf-dim one requires also that the space is complete).

example: photon polarization is described by vectors in $\mathbb{C}^2$.

$$\text{Here} \quad \mathcal{H} = \mathbb{C}^2 = \{(\alpha, \beta) \mid |\alpha|^2 + |\beta|^2 = 1\}.$$

$$|\psi\rangle = \alpha|x\rangle + \beta|y\rangle \quad \text{in Dirac notation}$$

$$\text{where} \quad |\alpha|^2 + |\beta|^2 = 1 \quad ; \alpha, \beta \in \mathbb{C}.$$

- linear polarization $\quad |\theta\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle$
- circular polarization $\quad \cos\theta\,|x\rangle + i\sin\theta\,|y\rangle$
- elliptic pol $\quad \cos\theta\,|x\rangle + e^{i\delta}\sin\theta\,|y\rangle \; ; \; \delta \neq 0$

b) State vectors $|\psi\rangle$ evolve with time. The time evolution is a unitary map:

$$U(t): \mathcal{H} \longrightarrow \mathcal{H}$$

$$|\psi\rangle \longmapsto U(t)|\psi\rangle = \text{state at time } t.$$

We have $U(t=0) = \mathbb{1}$ ; $U(t)\,U(t') = U(t+t')$.

[Usually one computes $U(t)$ from Schrödinger's equation:

$$i\hbar \frac{d}{dt} U(t) = H(t)\, U(t)$$

where $H(t)$ is the Hamiltonian. In this course we do not need to know what are the Hamiltonians; we will directly consider appropriate unitary evolutions.]

example: photon state evolves as $e^{i(kz-\omega t)}|\psi\rangle$ for a monochromatic photon of frequence $\omega$. Here $\omega = ck$ ($c$ = speed of light).

## c) Measurement postulate: first encounter.

Let $|\psi\rangle$ be the state of the system and suppose we perform a measurement at time $t$. The probability that we observe the state $|\varphi\rangle$ just after the measurement $(t + 0_+)$ is

$$|\langle \varphi | \psi \rangle|^2 .$$

One says that the state has been "reduced" from $|\psi\rangle$ to $|\varphi\rangle$ by the measurement process. Another terminology for "state" is "wave function" and for "state reduction" is "collapse of wave function".

example: Suppose a photon is prepared in state $|\theta\rangle$.
Do a measurement with an analyser and a detector.
Prob to obs pol $\alpha$ $= |\langle \alpha | \theta \rangle|^2$.

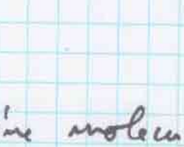
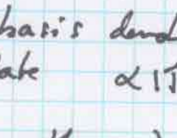
Remarks:
- States are defined up to a global phase $e^{i\lambda}|\psi\rangle$ because $e^{i\lambda}$ cancels out in all probability calculations. So state are really rays of the Hilbert space.
- States change in two very different (as far as we know) ways:
  - $\rightarrow$ unitary evolution
  - $\rightarrow$ measurement process (state reduction).

## 4) Notion of Quantum Bit. (Qubit or Qbit.).

We saw that the photon polarization lives in the space $\mathbb{C}^2$.
There are two orthonormal basis vectors that we can choose at will.
For example $\{|x\rangle, |y\rangle\}$ or $\left\{\frac{1}{\sqrt{2}}(|x\rangle+|y\rangle), \frac{1}{\sqrt{2}}(|x\rangle-|y\rangle)\right\}$.

In general quantum systems whose state vector lives in $\mathbb{C}^2$ are called two-level systems. Remarquably many such systems exist in nature. For example;

* photon polarization : $\alpha|x\rangle + \beta|y\rangle$
* electron spin, ion spin, atom spin for some ions and atoms. basis denoted $\{|\uparrow\rangle, |\downarrow\rangle\}$. general state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$.
* Benzene molecule (and many others) can be in states:

 or 

where — are chemical bonds involving a pair of electrons
= " " " two pairs of electrons.

These molecules can be in "resonating states":

$\alpha \,|$  $\rangle + \beta \,|$  $\rangle$

* A quantum bit is such a two level system or rather the abstraction of it. A quantum bit is described by a state vector of $\mathbb{C}^2$. The canonical basis is denoted by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A general state is

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle. \quad , \quad |\alpha|^2 + |\beta|^2 = 1.$$

* The difference with a classical bit which takes values $\{0, 1\}$ is that a quantum bit can exist in a "general superposition" of $|0\rangle$ and $|1\rangle$.

* If we try to access the information in $|\psi\rangle$ we have to make a measurement. Once we make an appropriate measurement the state will collapse in state $|0\rangle$ with prob $|\alpha|^2$ and in state $|1\rangle$ with prob $|\beta|^2$. However we do not have access to $\alpha$ and $\beta$ if we have only one such state because once the measurement has been performed the state vector is $|0\rangle$ or $|1\rangle$.

* The difference with classical bits as we will see lies in the power that the superposition principle gives to manipulation of states.

* Note that one cannot say that $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is in state $|0\rangle$ with prob $|\alpha|^2$ and state $|1\rangle$ with prob $|\beta|^2$. State $|\psi\rangle$ is just in state $|\psi\rangle$. Indeed we can write for example $|\psi\rangle$ in another basis:

$$|\psi\rangle = \frac{(\alpha+\beta)}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) + \frac{(\alpha-\beta)}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle).$$

There exists an appropriate measurement which collapse $|\psi\rangle$ to

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \text{ with prob } \frac{|\alpha+\beta|^2}{2} \\ \text{and } \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \text{ with prob } \frac{|\alpha-\beta|^2}{2}. \end{array}\right\}.$$

* With photons measurement in the $\{|0\rangle, |1\rangle\}$ basis corresponds to pass photons through the analyser in the $\{|x\rangle, |y\rangle\}$ basis

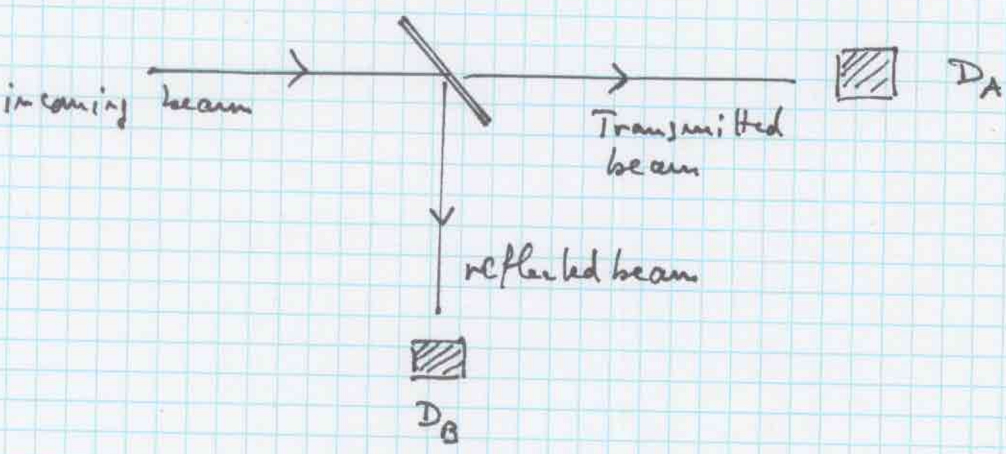
whereas measurement in the $\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right)$ basis corresponds to pass photons through the analyser in the $\{|\alpha=45°\rangle, |\alpha=135°\rangle\}$ basis

# 5) A flavor of two quantum applications.

## a) Random Number Generator.

With QM one can generate binary sequences that are intrinsically random. Here is one way to do this (in principle):

Send a beam on a semi-transparent mirror which has the effect to split the beam of light in two equal intensity beams: the transmitted and reflected beams.




We model this situation with a two dimensional Hilbert space for the photon's momentum: horizontal $|H\rangle$ and vertical $|V\rangle$.

- The initial state is $|H\rangle$.
- After the beam splitter it is $\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$.
- If $D_A$ clicks we collect a 1. If $D_B$ clicks we collect a 0. Sending photons one by one results in a sequence of 0 and 1: 0 1 0 1 0 0 1 1 0 1 0 0 0 1 1 0 1 0 1

- According to quantum mechanics this sequence is perfectly random

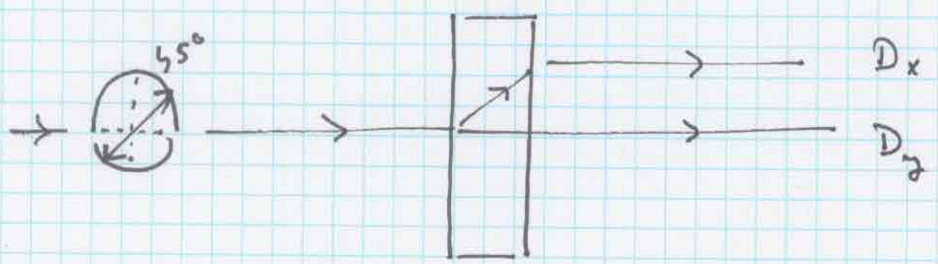
$$\text{Prob}(1) = \left| \langle H| \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \right|^2 = \frac{1}{2} \left| \langle H|H\rangle + \langle H|V\rangle \right|^2$$
$$= \frac{1}{2}.$$

$$\text{Prob}(0) = \left| \langle V| \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \right|^2 = \frac{1}{2}.$$

- One could arrive at the same conclusion by preparing a $45^\circ$ polarized photons

$$|45^\circ\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$$

and send them one by one through a PBS




We record 1 if $D_y$ clicks and 0 if $D_x$ clicks.

Remarks: Such principles have been put to work in labs and there is an existing technology. For example Id Quantique - Geneva.

## b) Quantum Parallelism.

We briefly illustrate the notion of quantum parallelism, a notion to which we will come back in part III. One can manipulate Qbits just as one can manipulate classical bits.

- For example the classical NOT gate takes a bit and negates it:

$$b \longrightarrow \boxed{\text{NOT}} \longrightarrow \bar{b} \qquad \text{or} \qquad \bar{b} = b \oplus 1.$$

$$b = 0 \text{ or } 1 \qquad\qquad \bar{b} = 1 \text{ or } 0.$$

Quantum mechanicaly the input is a quantum bit

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

and the output is

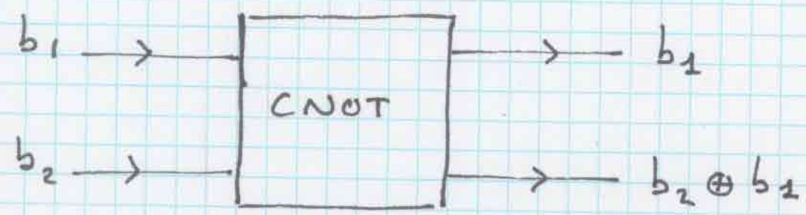
$$|\psi'\rangle = \alpha\,|1\rangle + \beta\,|0\rangle$$

The swep of 0 and 1 is a unitary operation: the quantum NOT gate. Indeed set $\text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We see that

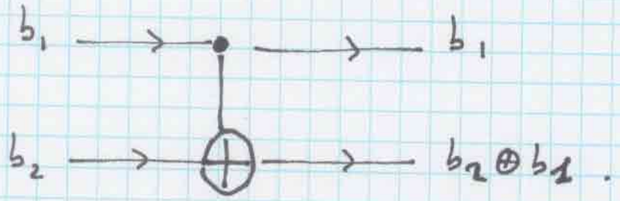
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so by linearity $\text{NOT}\,|\psi\rangle = |\psi'\rangle$. $\text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a unitary matrix so in principle a device can be constructed that unitarily evolves $|\psi\rangle$ to $|\psi'\rangle$ (with time).

- Another important classical gate is the CNOT "control not". There we have two bits, the first one being called the control bit. If the control bit $b_1 = 1$, the second bit $b_2$ is negated while if the control bit $b_2 = 0$, the second bit is unchanged:




sometimes also drawn as




The quantum version is the following operation

input:

$$|\psi\rangle = \sum_{b_1, b_2} \alpha_{b_1 b_2} |b_1, b_2\rangle = \alpha_{00} |0,0\rangle + \alpha_{01} |0,1\rangle + \alpha_{10} |1,0\rangle + \alpha_{11} |1,1\rangle$$

output

$$|\psi'\rangle = \sum_{b_1, b_2} \alpha_{b_1 b_2} |b_1, b_2 \oplus b_1\rangle = \alpha_{00} |0,0\rangle + \alpha_{01} |0,1\rangle + \alpha_{10} |1,1\rangle + \alpha_{11} |1,0\rangle$$

Note that here the quantum states live in $\mathcal{H} = \mathbb{C}^4 \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$.

It is possible to check that CNOT is a unitary operation or matrix. The easiest way is to note that it is linear and preserves the norm. Thus a physical device can (in principle) be constructed that performs the operation:

$$\text{CNOT}\left(\sum_{b_1,b_2} \lambda_{b_1 b_2} |b_1, b_2\rangle\right) = \sum_{b_1,b_2} \lambda_{b_1 b_2} |b_1, b_2 \oplus b_1\rangle .$$

* Here our physical device performs four operations in the second sum in parallel. This is called "quantum parallelism" and is exploited in quantum computation (part III).

* Note however that the accessible information does not correspond to all four results $|b_1, b_2 \oplus b_1\rangle$ because a measurement in the $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ basis will reduce the state to one of these. Despite this it is sometimes possible to access some global information about the function we compute. (see exercises).