Lecture 1.

1) Experiments with polarizers: electromagnetic waves.

2) Exps with polarizers: photons.

3) First set of quantum mechanical rules

4) Two level systems and Qbit.

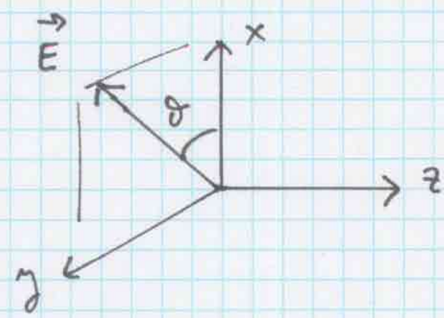
5) A flavor of quantum applications: random number generator and CNOT gate, quantum parallelism.

1) Experiments with polarizers: electromagnetic waves.

Light is an electromagnetic wave of electric $\vec{E}$ and magnetic $\vec{B}$ fields in vacuum. If the wave propagates along the z-axis, $\vec{E} \perp \vec{B}$ and $(\vec{E}, \vec{B}) \in$ (x-y plane). Since $\vec{E}$ and $\vec{B}$ are related by Maxwell's equations we need only look at the electric field $\vec{E}$. Consider a monochromatic mode of frequency ω propagating along z. The electric field can be represented as a complex vector $\vec{E} = (E_x, E_y, 0)$,

$$\begin{cases} E_x = E_0 \cos\vartheta \; e^{i(kz-\omega t)} \\ E_y = E_0 \sin\vartheta \; e^{i(kz-\omega t)} \cdot e^{i\delta} \end{cases}$$

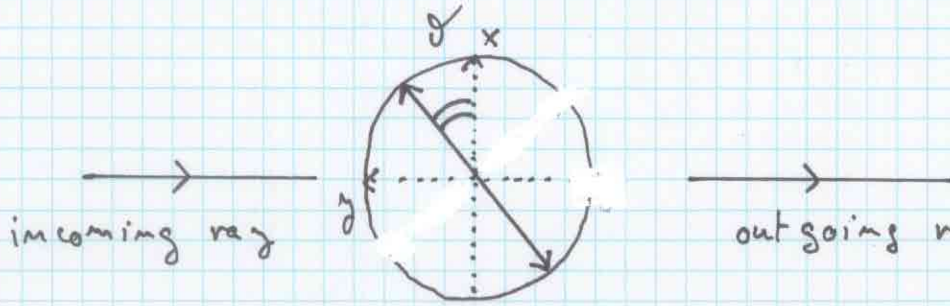
- For $\delta_x = \delta_y$ the polarization is linear: $\vec{E}$ oscillates along an axis at an angle ϑ with x (in the xy plane)



- For $\delta_x - \delta_y = \frac{\pi}{2}$ the polarization is circular: $\vec{E}$ describes a circle as a function of time in the x-y plane.

- For $\delta \neq 0$ in general the polarization is elliptical: $\vec{E}$ describes an ellipse in the x-y plane.

Linearly polarized light can be prepared by a polarizer with axis oriented along ϑ. The effect of the polarizer is to select the components of $\vec{E}$ along the axis ϑ.



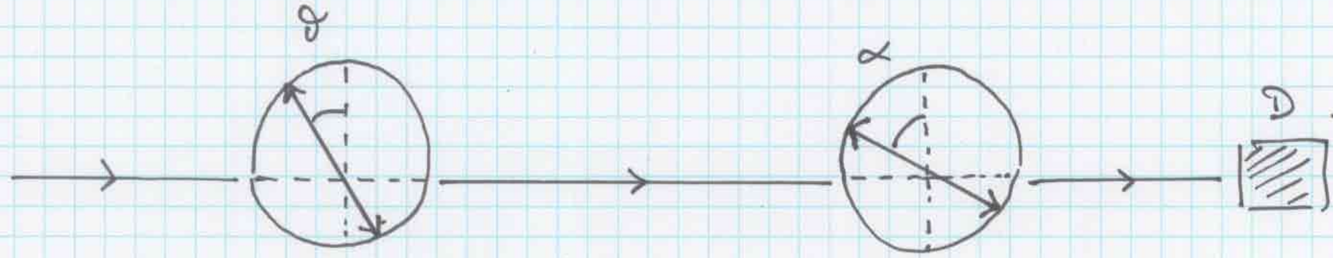
$$\vec{E} = (E_0 \cos\vartheta, E_0 \sin\vartheta, 0)\, e^{i(kz-\omega t)}.$$

Polariser - Analyser Experiment.

The intensity of the outgoing wave is

$$I = |\vec{E}|^2 = E_0^2 \cos^2\vartheta + E_0^2 \sin^2\vartheta = E_0^2 .$$

Suppose that now we pass the outgoing ray through a second polarizer with an axis at an angle α :



What is the intensity detected in detector D? After polariser α (called an analyzer in this context) the electric field is

$$\vec{E}' = (\vec{E} \cdot \vec{e}_\alpha)\, \vec{e}_\alpha = E_0 \cos(\vartheta - \alpha)\, e^{i(kz-\omega t)} \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}$$

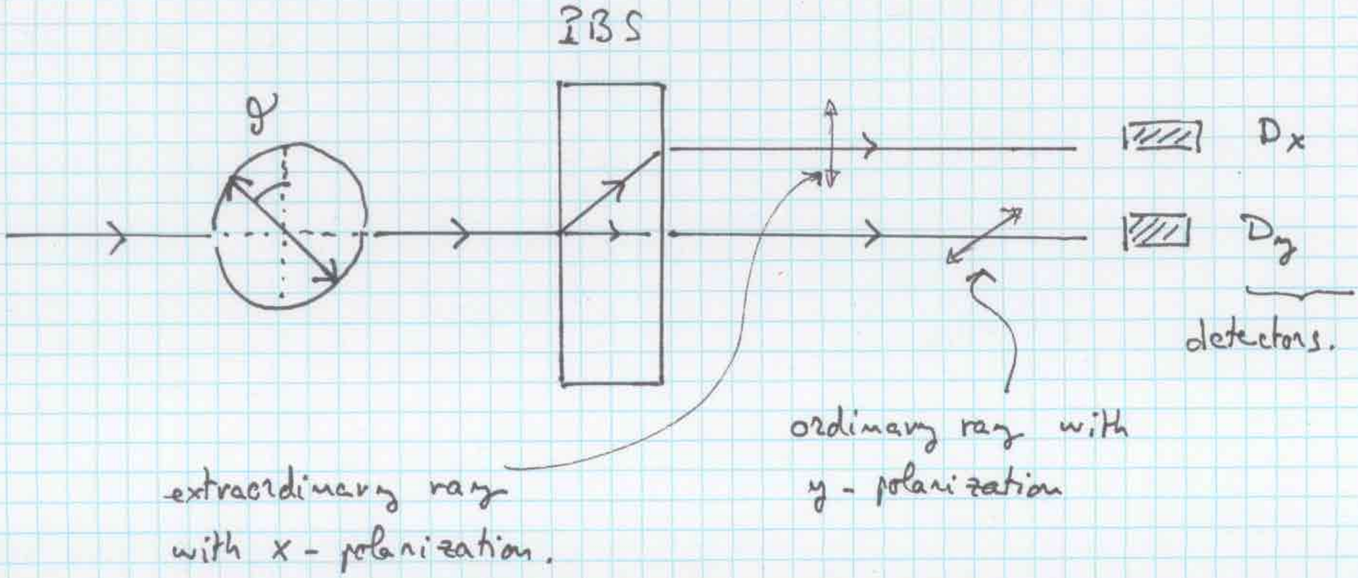
So the intensity detected in D is

$$I' = E_0^2 \cos^2(\vartheta - \alpha) \quad .$$

If $\alpha \parallel \vartheta$ all the light passes through the second polarizer while if $\alpha \perp \vartheta$ all the light is absorbed in the second polarizer. [in fact this experiment demonstrates the vectorial nature of e.m waves or light.].

Decomposition Exp. with a PBS.

Here PBS stands for "polarizing beam splitter". Calcite crystals have the property of splitting a linearly polarized beam into two beams: the "ordinary" ray with horizontal polarisation and the "extraordinary" ray with vertical polarisation.



- Between the polarizer and the PBS the electric field is

$$\vec{E} = E_0 \begin{pmatrix} \cos\vartheta \\ 0 \\ 0 \end{pmatrix} e^{i(kz-\omega t)} + E_0 \begin{pmatrix} \\ \sin\vartheta \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

The intensity is $I = E_0^2$.

- For the extra-ordinary ray the electric field is

$$E_0 \begin{pmatrix} \cos\vartheta \\ 0 \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

and the intensity at D_y is $I_y = E_0^2 \cos^2\vartheta$

- For the ordinary ray the electric field is

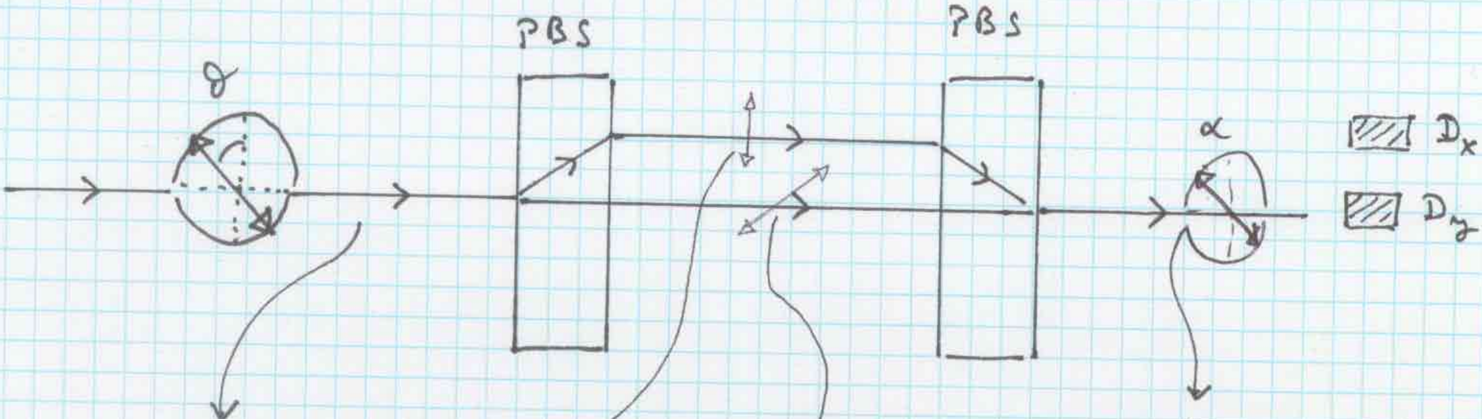
$$E_0 \begin{pmatrix} 0 \\ \sin\vartheta \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

and the intensity at D_x is $I_x = E_0^2 \sin^2\vartheta$.

(of course $I_x + I_y = I$).

Decomposition - Recombination Exp with two PBS.

If we decompose light with a PBS we can also recombine it with a symmetric calcite crystal, i.e a symmetric PBS. The picture of the exp. is :



$$\vec{E} = E_0 \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{pmatrix} \qquad \vec{E}_x = E_0 \begin{pmatrix} \cos\vartheta \\ 0 \\ 0 \end{pmatrix} \qquad \vec{E}_y = E_0 \begin{pmatrix} 0 \\ \sin\vartheta \\ 0 \end{pmatrix} \qquad \vec{E}' = \left[(\vec{E}_x + \vec{E}_y) \cdot \vec{e}_\alpha\right] \cdot \vec{e}_\alpha$$

- The intensity measured in D_y is $|\vec{E}'|^2 = |(\vec{E}_x + \vec{E}_y) \cdot \vec{e}_\alpha|^2 = E_0^2 \cdot \cos^2(\vartheta - \alpha)$
- The intensity measured in D_x is 0.

After the second PBS the total electric field is the linear superposition of $\vec{E}_x$ and $\vec{E}_y$. First we superpose the electric fields and then we square the total field to find the total intensity. This is a manifestation of the wave nature of light.

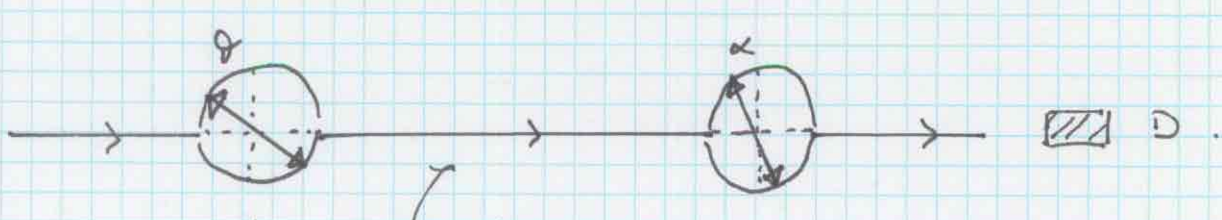
2) Exps with polarizers: photons.

The concept of photon emerged at the beginning of the 20th century from work of Planck (1900) and Einstein (1905) that attempted to explain thermodynamic properties of radiation in terms of statistical mechanics (itself a newly founded branch of science).

If we diminish sufficiently the intensity of a light beam we arrive at a situation where photons propagate "one by one".
We will repeat the exps of last paragraph with single photons.

Polarizer - Analyzer pair exp.

We take a polarizer - analyzer pair and suppose there is at each instant of time one photon between them:



photons pass one by one

- If $\vartheta = \alpha$: detector D clicks always (photons observed always)
- If $\vartheta \perp \alpha$: detector D does not click (no photon observed).
- General $\vartheta \neq \alpha$: sometimes D clicks, sometimes not. The sequence of clicks is purely random 1 0 0 1 1 0 1 0 1 0 0 0 1 0 0 0 0 0 1 ...

Frequency of clicks $\propto \cos^2(\vartheta - \alpha)$.

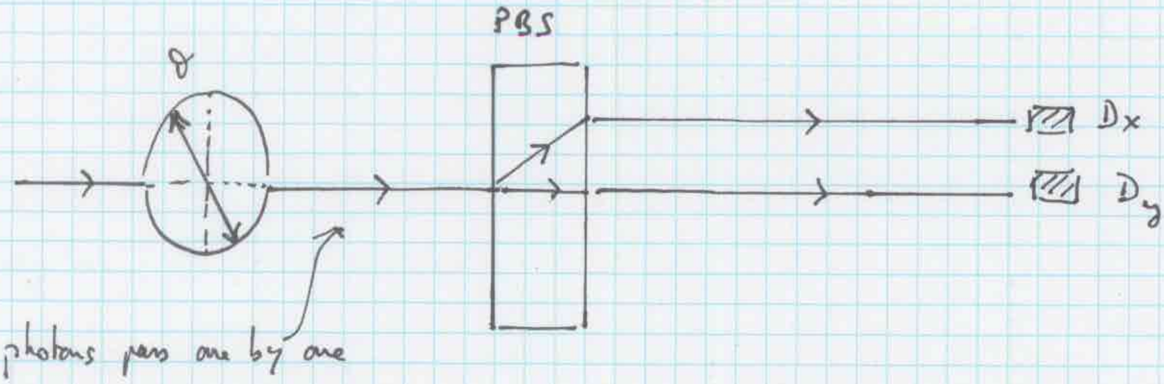
* So the probability that a photon with polarisation along ϑ passes through polarizer at angle α is $\cos^2(\vartheta - \alpha)$.

This is compatible with the intensity of e.m wave which also was $\cos^2(\vartheta - \alpha)$.

* However there is a "problem": photon is indivisible particle so that when it is detected in D its polarization must be along α.

The polarization state has changed from ϑ to α and this is quite different than the electric field of the e.m wave. For the e.m wave the component perpendicular to α is absorbed.

Decomposition Exp with a PBS.



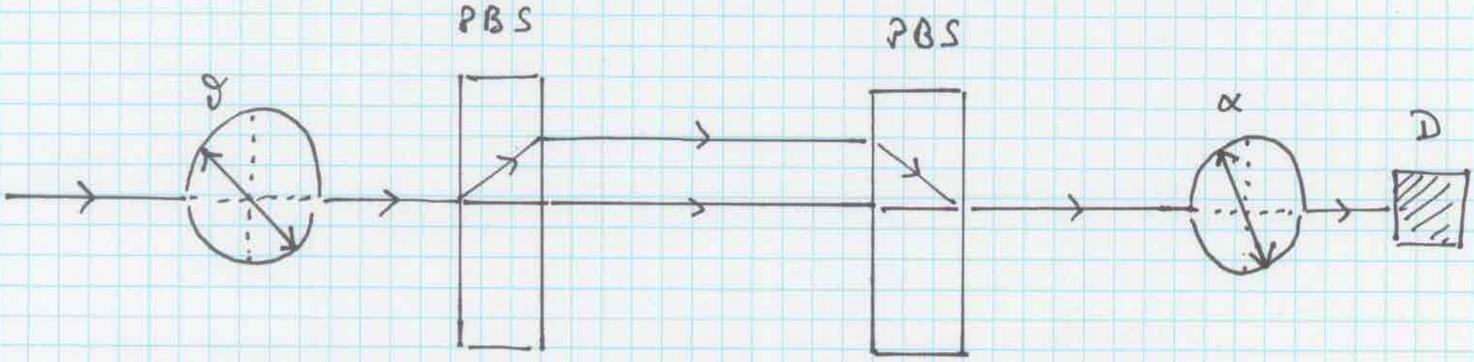
- A photon is detected either at D_x or at D_y.
- looking at the sequence of clicks we find that

$$\text{Prob (click at } D_y) = \sin^2\vartheta \; ; \quad \text{Prob (click at } D_x) = \cos^2\vartheta$$

The "classical" interpretation would say that since the photon is an indivisible particle it has a probability $\sin^2\vartheta$ (resp $\cos^2\vartheta$) to take the upper path (resp lower path). However the next experiment will show that this interpretation is wrong.

Note that the above probabilities are consistent with intensities predicted by the wave theory.

Decomposition - Recombination Exp.



- According to wave theory the intensity observed at D is

$$\cos^2(\vartheta - \alpha).$$

Even though light is decomposed into horizontally and vertically polarized beams by the first PBS, this has no influence on the end result since it is recombined by the second PBS.

– Now suppose we send photons one by one.

"Classical prediction"

Let us first use standard probabilistic reasoning to predict the probability of detection at D.

- If photon takes lower path its polarization is horizontal (along y). Then

$$\text{Prob (click at D | lower path)} = \cos^2\left(\frac{\pi}{2} - \alpha\right) = \sin^2\alpha .$$

- If photon takes upper path its polarization is vertical (along x). Then

$$\text{Prob (click at D | upper path)} = \cos^2(0 - \alpha) = \cos^2\alpha .$$

So according to usual probabilistic rules:

$$\begin{aligned}\text{Prob (click at D)} &= \text{Prob (click at D | lower path) Prob (lower path)} \\ &+ \text{Prob (click at D | upper path) Prob (upper path)}\end{aligned}$$

$$\Rightarrow \underline{\text{Prob (click at D)} = \sin^2\alpha \sin^2\vartheta + \cos^2\alpha \cos^2\vartheta} .$$

This is not consistent with wave theory which predicts an intensity

$$\cos^2(\vartheta - \alpha) = \cos^2\alpha \cos^2\vartheta + \sin^2\alpha \sin^2\vartheta + \underbrace{2 \sin\alpha \sin\vartheta \cos\alpha \cos\vartheta}_{\frac{1}{2}\sin 2\alpha \sin 2\vartheta}$$

The third term which is missing is precisely the interference term obtained because of the superposition principle for waves (or for the electric field).

"quantum prediction"

We must find a way to describe single photons by "states" that can "interfere" much like waves do. Yet we would like to view them as indivisible quanta of energy.

- Photons have "polarization state vectors" (basically this is the state of their electric field). A polarization along ϑ is described by $(\exp i(kz-\omega t)).\vec{e}_\vartheta$ where $\vec{e}_\vartheta$ is the unit vector along ϑ. We will immediately conform to Dirac's notation an set

$$\vec{e}_\vartheta = |\vartheta\rangle$$

the "Ket ϑ". The hermitian transposed vector $\vec{e}_\vartheta^{\,T*}$ is denoted by

$$\langle\vartheta| = \vec{e}_\vartheta^{\,T*}$$

the "bra ϑ".

The inner product is

$$\vec{e}_\alpha^{\,T*} . \vec{e}_\vartheta = \langle\alpha|\vartheta\rangle$$

the "bracket".

So after the first polarizer at an angle ϑ the photon has been prepared in the state vector.

$$e^{i(kz-\omega t)} \quad |\vartheta\rangle$$

We shall drop the phase in what follows because it plays no role.

As suggested by the denomination "state vector" these form a vector space. Here the vector space is two dimensional: we can take $|x\rangle$, $|y\rangle$ as basis vectors where $|x\rangle$ is "vertical polarization" and $|y\rangle$ "horizontal polarization".

So

$$|\vartheta\rangle = \cos\vartheta \, |x\rangle + \sin\vartheta \, |y\rangle$$

This vector space has a natural scalar product. In Dirac's notation the scalar product of $|\alpha\rangle$ and $|\vartheta\rangle$ is

$$\langle\alpha|\vartheta\rangle = \left(\cos\alpha \, \langle x| + \sin\alpha \, \langle y|\right)\left(\cos\vartheta \, |x\rangle + \sin\vartheta \, |y\rangle\right)$$

$$= \cos\alpha\cos\vartheta \, \langle x|x\rangle + \cos\alpha\sin\vartheta \, \langle x|y\rangle + \sin\alpha\cos\vartheta \, \langle y|x\rangle + \sin\alpha\sin\vartheta \, \langle y|y\rangle$$

$$\Rightarrow \langle\alpha|\vartheta\rangle = \cos\alpha\cos\vartheta + \sin\alpha\sin\vartheta = \cos(\vartheta-\alpha).$$

Let us now try to reinterpret the three experiments with state vectors of single photons :

Polarizer - Analyzer pair :

- After first polarizer state of photon is $|\vartheta\rangle$.
- After second polarizer if photon is detected state vector is $|\alpha\rangle$.

(if photon is not detected it has been absorbed by the polarizer) .

- Repeating this exp many times we conclude that

$$\text{Prob (photon is detected in state } |\alpha\rangle \text{)} = \cos^2(\vartheta - \alpha) = |\langle\alpha|\vartheta\rangle|^2 .$$

This is our first encounter with the "measurement postulate".

The probability that the polarization is measured along α is $|\langle\alpha|\vartheta\rangle|^2$ and just after the measurement the state is reduced to $|\alpha\rangle$ (before it was $|\vartheta\rangle$).

Decomposition with PBS exp.

- Before PBS the state is $|\vartheta\rangle$. This can also be expressed as $|\vartheta\rangle = \cos\vartheta\, |x\rangle + \sin\vartheta\, |y\rangle$.
- After PBS the state has x - polarization on the upper path .

y - polarization on the lower path .

It is

$$|\psi\rangle = \cos\vartheta\, |x, u\rangle + \sin\vartheta\, |y, l\rangle .$$

- Photon in state $|\psi\rangle$ will click either at D_x or at D_y.
  [There is only one photon in the game !].

- Repeating the experiment we find that

$$\text{Prob (click at } D_x) = \cos^2\vartheta = |\langle x, u | \psi\rangle|^2$$

$$\text{Prob (click at } D_y) = \sin^2\vartheta = |\langle y, l | \psi\rangle|^2 .$$

Decomposition - reconstitution exp with two PBS.

- After first pol at angle ϑ the state is $|\vartheta\rangle$.

- After first PBS state is $(\cos\vartheta\, |x, u\rangle + \sin\vartheta\, |y, l\rangle)$

- After second PBS state is $(\cos\vartheta\, |x\rangle + \sin\vartheta\, |y\rangle)$ since the upper and lower paths have merged again.

- After analyser at angle α if photon is detected it is in state $|\alpha\rangle$.

- Repeating the experiment the probability the pol α is observed is

$$\text{Prob (pol } \alpha \text{ is observed)} = |\langle\alpha| (\cos\vartheta\, |x\rangle + \sin\vartheta\, |y\rangle)|^2 = \cos^2(\vartheta - \alpha).$$