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Einselection and Decoherence from an Information Theory Perspective

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Abstract.
We introduce and investigate a simple model of conditional quantum dynamics. It allows for a discussion of the information-theoretic aspects of quantum measurements, decoherence, and environment-induced superselection (einselection).

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1 Introduction

Transfer of information was the focus of attention [1-3] of research on decoherence since the early days. In the intervening two decades this perspective was not forgotten [4], but the study of different mechanisms of decoherence [5-9] took precedence over considerations of information-theoretic nature. The aim of this paper is to sketch a few ideas which tie the "traditional" points of view of einselection and decoherence (especially the issue of the preferred pointer basis) to various other aspects of decoherence that have a strong connection with information-theoretic concepts.

A large part of our discussion shall be based on a simple model of conditional dynamics, which is a direct generalization of the "bit by bit" measurement introduced in [1] and studied in [3]. We shall introduce the model in Section 2 and use it to compute the "price" of information in units of action in Section 3. Section 4 defines information theoretic quantum *discord* between two classically identical definitions of mutual information. Discord can be regarded as a measure of a violation of classicality of a joint state of two quantum subsystems. Section 5 turns to the evolution of the state of the environment in course of decoherence. The *redundancy ratio* introduced there can be regarded as a measure of objectivity of quantum states. A large redundancy ratio is a sufficient condition for an effective classicality of quantum states.

2 Controlled shifts for conditional dynamics

The simplest example of an entangling quantum evolution is known as the controlled not (`c-not`). It involves two bits (a "control" and a "target"). Their interaction leads to:

$$|0_C\rangle|x_T\rangle \longrightarrow |0_C\rangle|x_T\rangle \tag{1a}$$

$$|1_C\rangle|x_T\rangle \longrightarrow |1_C\rangle|\neg x_T\rangle \tag{1b}$$

where the state $|\neg x\rangle$ is defined through a basis-dependent negation:

$$\neg(\gamma|0_T\rangle + \eta|1_T\rangle) = \gamma|1_T\rangle + \eta|0_T\rangle \ . \tag{2}$$

Classical `c-not` "flips" the target bit whenever the control is in the state "1", but does nothing otherwise. Quantum `c-not` is an obvious generalization.

The distinction between the classical and quantum `c-not` comes from the fact that both quantum and classical bits can be in an arbitrary superposition. Thus, `c-not` starting from a superposition of $|0\rangle$ and $|1\rangle$ will in general lead to an entangled state. Moreover, when both the control and the target start in Hadamard-transformed states:

$$|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2} \ , \tag{3}$$

`c-not` reverses direction:

$$|\pm\rangle|+\rangle \longrightarrow |\pm\rangle|+\rangle; \tag{4a}$$

$$|\pm\rangle|-\rangle \longrightarrow |\mp\rangle|-\rangle \ . \tag{4b}$$

Above, we have dropped labels: The original control is always to the left, as was the case in Eq. (1). We say "the original", because the Hadamard transform of Eq. (3) reverses the direction of the information flow in the quantum `c-not`. As can be seen in Eq. (4), the sign of the former control (left ket) flips when the former target is in the state $|-\rangle$.

Study of such simple models has led to the concept of preferred pointer states [1] and einselection [2,3]. We shall not review here these well known developments, directing the reader instead to the already available [10] or forthcoming [11] reviews of the subject.

Controlled shift (`c-shift`) is a straightforward generalization of `c-not`. The original truth table (an analogue of Eq. (1)) can be written as:

$$|s_j\rangle|A_k\rangle \longrightarrow |s_j\rangle|A_{k+j}\rangle \tag{5}$$

There is also a control and a target (which we shall more often call "the system $\mathcal{S}$" and "the apparatus $\mathcal{A}$", reflecting this nomenclature in notation). Equation (5) implies Eq. (1) when both $\mathcal{S}$ and $\mathcal{A}$ have two-dimensional Hilbert spaces. Moreover, when $j = 0$, Eq. (1) becomes a model of a pre-measurement:

$$|\Psi_0\rangle = |\psi\rangle|A_0\rangle = \left(\sum_i a_i|s_i\rangle\right)|A_0\rangle \longrightarrow \sum_i a_i|s_i\rangle|A_i\rangle = |\Psi_t\rangle \ . \tag{6}$$

As in the case of c-not, in the respective bases $\{|s_i\rangle\}$ and $\{|A_k\rangle\}$, Eq. (5) seems to imply a one - way flow of information, from $\mathcal{S}$ to $\mathcal{A}$. However, a complementary basis [12,13] can be readily defined:

$$|B_k\rangle \; = \; N^{-\frac{1}{2}} \sum_{l=0}^{N-1} \exp(\frac{2\pi i}{N} kl) \; |A_l\rangle \; . \tag{7}$$

It is analogous to the Hadamard transform we have introduced before, but it also has an obvious affinity to the Fourier transform. We shall call it a Hadamard-Fourier Transform (HFT). It is straightforward to show that the orthonormality of $\{|A_k\rangle\}$ immediately implies:

$$\langle B_l|B_m\rangle = \delta_{lm} \; . \tag{8}$$

The inverse of HFT can be easily given:

$$|A_k\rangle \; = \; N^{-\frac{1}{2}} \sum_{l=0}^{N-1} \exp(-\frac{2\pi i}{N} kl) \; |B_l\rangle \; . \tag{9}$$

Consequently, for an arbitrary $|\psi\rangle$;

$$|\psi\rangle = \sum_n \alpha_n |A_n\rangle = \sum_k \beta_k |B_k\rangle \; , \tag{10}$$

where the coefficients are given by the HFT;

$$\beta_k = N^{-\frac{1}{2}} \sum_{n=0}^{N-1} \exp(-\frac{2\pi i}{N} kn) \alpha_n . \tag{11}$$

To implement the truth table of Eq. (5) we shall use observables of the apparatus:

$$\hat{A} = \sum_{k=0}^{N-1} k |A_k\rangle\langle A_k| ; \tag{12a}$$

$$\hat{B} = \sum_{l=0}^{N-1} l |B_l\rangle\langle B_l| \; , \tag{12b}$$

as well as the observable of the system:

$$\hat{s} = \sum_{l=0}^{N-1} l |s_l\rangle\langle s_l| \; . \tag{13}$$

The interaction Hamiltonian of the form;

$$H_{int} = g \hat{s} \hat{B} \tag{14}$$

acting over a period t will induce a transition;

$$\exp(-iH_{int}t/\hbar)|s_j\rangle|A_k\rangle = |s_j > N^{-\frac{1}{2}} \sum_{l=0}^{N-1} \exp[-i(jgt/\hbar + 2\pi k/N)l]|B_l\rangle \; . \tag{15}$$

Thus, if the coupling constant g is selected so that the associated action is

$$I = gt/\hbar = G \times 2\pi/N \ . \tag{16}$$

a perfect one-to-one correlation between the states of the system and of the apparatus can be accomplished, since:

$$\exp(-iH_{int}t/\hbar)|s_j\rangle|A_k\rangle \ = \ |s_j\rangle|A_{\{k+G*j\}_N}\rangle \ . \tag{17}$$

Thus, true to its name, the interaction described here accomplishes a simple shift, Eq. (5), of the state of the apparatus, while the system acts as a control. The index $\{k+G*j\}_N$ has to be evaluated modulo N (where N is the number of the orthogonal states of the apparatus) so that when $k+G*j > N$, the apparatus states can "rotate" through $|A_0\rangle$ and stop where the interaction takes it. The integer G can be regarded as the gain factor. As Eqs. (14) - (17) imply, the adjacent states of the system (i.e., $|s_j\rangle, \ |s_{j+1}\rangle$) get mapped onto the states of the apparatus that are G apart "on the dial".

When the dimension of the Hilbert space of the system n is such that

$$nG < N \ , \tag{18}$$

the above model provides one with a simple example of amplification. It is possible to use it to study the utility of amplification in increasing signal to noise ratio in measurements [11]. It also shows why amplification can bring about decoherence and effective irreversibility (although `c-shift` is of course perfectly reversible). We shall employ `c-shift` to study the cost of information transfer, to introduce information – theoretic discord, the measure of the classicality of correlations, and to discuss objectivity of quantum states which arises from the redundancy of the records imprinted by the state of the system on its environment.

3 Planck's constant and the price of a bit

Transfer of information is the objective of the measurement process and an inevitable consequence of most interactions. It happens in course of decoherence. Here we shall quantify its cost in the units of action.

The consequence of the interaction between $\mathcal{S}$ and $\mathcal{A}$ is the correlated state $|\Psi_t\rangle$, Eq. (6). While the joint state of $\mathcal{AS}$ is pure, each of the subsystems is in a mixed state given by the reduced density matrix of the system

$$\rho_{\mathcal{S}} \ = \mathrm{Tr}_{\mathcal{A}}|\Psi_t\rangle\langle\Psi_t| \ = \ \sum_{i=0}^{N-1} |a_i|^2|s_i\rangle\langle s_i| \ ; \tag{19a}$$

and of the apparatus

$$\rho_{\mathcal{A}} \ = \mathrm{Tr}_{\mathcal{S}}|\Psi_t\rangle\langle\Psi_t| \ = \ \sum_{i=0}^{N-1} |a_i|^2|A_i\rangle\langle A_i| \ . \tag{19b}$$

The correlation brought about by the interaction, Eq. (6), leads to the loss of information about $\mathcal{S}$ and $\mathcal{A}$ individually. The entropy of each increases to

$$\mathcal{H}_{\mathcal{S}} = -\mathrm{Tr}\rho_{\mathcal{S}} \log \rho_{\mathcal{S}} = -\sum_{i=0}^{N-1} |a_i|^2 \log |a_i|^2 = -\mathrm{Tr}\rho_{\mathcal{A}} \log \rho_{\mathcal{A}} = \mathcal{H}_{\mathcal{A}} \ . \tag{20}$$

As the evolution of the whole $\mathcal{AS}$ is unitary, the entropies of the subsystems must be compensated by the decrease of their mutual entropy, i.e., by the increase of their mutual information:

$$\mathcal{I}(\mathcal{S}:\mathcal{A}) = \mathcal{H}_{\mathcal{S}} + \mathcal{H}_{\mathcal{A}} - \mathcal{H}_{\mathcal{SA}} = -2 \sum_{i=0}^{N-1} |a_i|^2 \log |a_i|^2 \tag{21}$$

Above, $H_{\mathcal{SA}}$ is the joint entropy of $\mathcal{SA}$. This quantity, Eq. (21), was introduced in the quantum context as a measure of entanglement some time ago [3] and has been since rediscovered and used [14].

The cost of a bit of information in terms of some other physical quantity is an often raised question. In the context of our model we shall inquire what is the cost of a bit transfer in terms of action. Let us then consider a transition represented by Eq. (6). The associated action must be no less than

$$I = \sum_{j=0}^{N-1} |a_j|^2 \arccos |\langle A_0 | A_j \rangle| \, . \tag{22}$$

When $\{|A_j\rangle\}$ are mutually orthogonal, the action is:

$$I = \pi/2 \tag{23a}$$

in Planck ($h = 2\pi\hbar$) units. This estimate can be lowered by using a judiciously chosen initial $|A_0\rangle$ which is a superposition of the outcomes $|A_j\rangle$. For a two-dimensional Hilbert space the average action can be thus brought down to $\pi\hbar/4$ [1,3]. In general, an interaction of the form

$$H_{\mathcal{SA}} = ig \sum_{k=0}^{N-1} |s_k\rangle\langle s_k| \sum_{l=0}^{N-1} (|A_k\rangle\langle A_l| - |A_l\rangle\langle A_k|) \tag{24}$$

saturates at the lower bound given by

$$I = \arcsin \sqrt{1 - 1/N} \ . \tag{23b}$$

As the dimensionality of the Hilbert spaces increases, the least action approaches $\pi/2$ per completely entangling interaction. The action per bit will be less when, for a given N, the transferred information is maximized, which happens when $|a_i|^2 = 1/N$ in Eq. (22). Then the cost of information in Planck units is

$$\iota = \frac{I}{\log N} \approx \frac{\pi}{2 \log_2 N} \ . \tag{25}$$

The cost of information per bit decreases with increasing N, the dimension of the Hilbert space of the smaller of the two entangled systems.

This result is at the same time both enlightening and disappointing: It shows that the cost of information transfer is not "fixed" (as one might have hoped). Rather, the least total amount of action needed for a complete entanglement is at least asymptotically fixed as Eqs. (23) show. Consequently, the least price per bit goes down when information is transferred "wholesale", i.e., when N is large. Yet, this is enlightening, as it may indicate why in the classical continuous world (where N is effectively infinite) one may be ignorant of that price and convinced that information is free.

4 Discord

Mutual information can be defined either by the symmetric formula, Eq. (21), or through an asymmetric looking equation which employs conditional entropy:

$$\mathcal{J}(\mathcal{S}:\mathcal{A}) = H_{\mathcal{S}} - H_{\mathcal{S}|\mathcal{A}} \tag{26}$$

Above, $H_{\mathcal{S}|\mathcal{A}}$ expresses the average ignorance of $\mathcal{S}$ remaining after the observer has found out the state of $\mathcal{A}$. In classical physics, the two formulae, Eqs. (21) and (26), are strictly identical, so that the *discord* between them:

$$\delta\mathcal{I} \;=\; \mathcal{I}(\mathcal{S}:\mathcal{A}) - \mathcal{J}(\mathcal{S}:\mathcal{A}) = 0 \tag{27a}$$

always disappears [15]:

$$\delta\mathcal{I} \;=\; 0\,. \tag{27b}$$

In quantum physics things are never this simple: To begin with, conditional information depends on the observer finding out about one of the subsystems, which implies a measurement. So $H_{\mathcal{S}|\mathcal{A}}$ must be carefully defined before Eqs. (26) and (27) become meaningful.

Conditional entropy is non-trivial in the quantum context because, in general, in order to find out $H_{\mathcal{S}|\mathcal{A}}$ one must choose a set of projection operators Π_j and define a conditional density matrix given by the outcome corresponding to Π_j through

$$\tilde{\rho}_{\mathcal{S}|\Pi_j} = \mathrm{Tr}_{\mathcal{A}}\Pi_j\rho_{\mathcal{S}\mathcal{A}}\,, \tag{28}$$

where in the simplest case $\Pi_j = |C_j\rangle\langle C_j|$ is a projection operator onto a pure state of the apparatus. A normalized $\rho_{\mathcal{S}|\Pi_j}$ can be obtained by using the probability of the outcome

$$p_j = \mathrm{Tr}\tilde{\rho}_{\mathcal{S}|\Pi_j}\ ; \tag{29}$$

$$\rho_{\mathcal{S}|\Pi_j} = p_j^{-1}\tilde{\rho}_{\mathcal{S}|\Pi_j}\ . \tag{30}$$

The conditional density matrix $\rho_{\mathcal{S}|\Pi_j}$ represents the description of the system $\mathcal{S}$ available to the observer who knows that the apparatus $\mathcal{A}$ is in a subspace defined by Π_j. For a pure initial state and an exhaustive measurement the conditional density matrix will also be pure. We shall however consider a broader range of possibilities, including joint density matrices which undergo a decoherence process, so that

$$\rho^P_{\mathcal{S}\mathcal{A}} = \sum_{i,j}\alpha_i\alpha_j^*|s_i\rangle\langle s_j||A_i\rangle\langle A_j| \Longrightarrow \sum_i|\alpha_i|^2|s_i\rangle\langle s_i||A_i\rangle\langle A_i| = \rho^D_{\mathcal{S}\mathcal{A}}\,. \tag{31}$$

This transition is accompanied by an increase in entropy

$$\Delta H(\rho_{\mathcal{SA}}) = H(\rho^D_{\mathcal{SA}}) - H(\rho^P_{\mathcal{SA}})$$

and by the simultaneous disappearance of the ambiguity in what was measured [1-3]. Now, $\rho_{\mathcal{S}|\Pi_j}$ is no longer pure, unless $\Pi_j = |A_j\rangle\langle A_j|$. That is, a measurement of the apparatus in bases other than the pointer basis will leave an observer with varying degrees of ignorance about the state of the system. More general cases when the density matrix is neither a pure pre-decoherence projection operator $\rho^P_{\mathcal{SA}}$ to the left of the arrow in Eq. (31) nor a completely decohered $\rho^D_{\mathcal{SA}}$ state on the right are possible and typical.

To define discord $\delta\mathcal{I}$ we finalize our definition of $H_{\mathcal{S}|\mathcal{A}}$:

$$H_{\mathcal{S}|\mathcal{A}} = \sum_i p_{|C_i\rangle} H(\rho^D_{\mathcal{S}|C_i\rangle}) \ . \tag{32}$$

Above, we have used an obvious notation for the density matrix conditioned upon pure states $\{|C_j\rangle\}$. We emphasize again that the conditional entropy depends on $\rho_{\mathcal{SA}}$, but also on the choice of the observable measured on $\mathcal{A}$. In classical physics all observables commute, so there is no such dependence. Thus, non-commutation of observables in quantum theory is the ultimate source of the information - theoretic discord.

The obvious use for the discord is to employ it as a measure of how non-classical the underlying correlation of two quantum systems is. In particular, when there exists a set of states in one of the two systems in which the discord disappears, the state represented by $\rho_{\mathcal{SA}}$ admits a classical interpretation of probabilities in that special basis. Moreover, unless the discord disappears for trivial reasons (which would happen in the absence of correlation, i.e., when $\rho_{\mathcal{SA}} = \rho_{\mathcal{S}}\rho_{\mathcal{A}}$), the basis which minimizes the discord can be regarded as "the most classical". For $\delta\mathcal{I} = 0$ the states of such preferred basis and their corresponding eigenvalues can be treated as effectively classical [11].

The vanishing discord is a stronger condition than the absence of entanglement. In effect, $\delta\mathcal{I} = 0$ implies existence of the eigenstates of $\rho_{\mathcal{SA}}$ which are products of the states of $\mathcal{S}$ and of $\mathcal{A}$. An instructive example of this situation arises as a result of decoherence: When the off-diagonal terms of $\rho_{\mathcal{SA}}$ disappear in a manner illustrated by Eq. (31), the discord disappears as well.

5 Environment as a witness: Redundancy ratio

The discussion of decoherence to date tends to focus on the effect of the environment on the system or on the apparatus. The destruction of quantum coherence and the emergence of preferred pointer observables whose eigenvalues are associated with the decoherence-free pointer subspaces was the focus of the investigation.

Here we shall break with this tradition. According to the theory of decoherence, the environment is monitoring the system. Therefore, its state must contain a record of the system. It is of obvious interest to analyze the nature and the role of this record. To this end, we shall use mutual information introduced before defining the *redundancy ratio*

$$\mathcal{R}_{\mathcal{I}(\otimes\mathcal{H}_{\mathcal{E}_k})} = \left(\sum_k \mathcal{I}(\mathcal{S} : \mathcal{E}_k) \right) / \mathcal{H}(\mathcal{S}) \quad . \tag{33}$$

Above, we imagined a setting where the system is decohering due to the interaction with the environment which is composed of many subsystems $\mathcal{E}_k$. $\mathcal{R}_{\mathcal{I}(\{\otimes\mathcal{H}_{\mathcal{E}_k}\})}$ is a measure of how many times – how redundantly – the information about the system has been inscribed in the environment. An essentially identical formula can be introduced using the asymmetric $\mathcal{J}$, Eq. (26). It is easy to establish that the discord is always non-negative and, hence, that

$$\mathcal{R}_{\mathcal{J}_{MAX}} \leq \mathcal{R}_{\mathcal{I}_{MAX}} \quad . \tag{34}$$

The subscript indicating maximization may refer to two distinct procedures: $\mathcal{R}_{\mathcal{J}}$ will obviously depend on the manner in which subsystems of the environment are measured. In fact, it is convenient to use

$$\mathcal{J}_k = \mathcal{J}(\mathcal{S} : \mathcal{E}_k) = \mathcal{H}(\mathcal{E}_k) - \mathcal{H}(\mathcal{E}_k|\mathcal{S}) \quad , \tag{35}$$

to define a basis-dependent

$$\mathcal{R}_{\mathcal{J}}(\{|s\rangle\}) = \mathcal{R}_{\mathcal{J}}(\otimes\mathcal{H}_{\mathcal{E}k}) \tag{36}$$

in a manner analogous to Eq. (33). Maximizing $\mathcal{R}_{\mathcal{J}}(\{|s\rangle\})$ with respect to the choice of the choice of states $\{|s\rangle\}$ [11] is an obvious “counterpoint” to the predictability sieve [16-19], the strategy which seeks states that entangle the least with the environment.

There is one more maximization procedure which may and should be considered: The environment can be partitioned differently – for example, it may turn out that more information about the system can be extracted by measuring, say, the photon environment in some collective fashion (homodyne?) instead of directly counting the environment photons. It is clear that for such optimization to be physically significant, it should respect to some degree the natural structure of the environment.

In addition to the redundancy ratio one can define the rate at which the redundancy ratio increases. The redundancy rate is defined as

$$\dot{\mathcal{R}} = \frac{d}{dt}\mathcal{R} \, . \tag{37}$$

Either the basis dependent or the basis-independent versions of $\dot{\mathcal{R}}$ may be of interest. The physical significance of the redundancy ratio rate is clear: It shows how quickly the information about the system spreads throughout the environment. It is, in effect, a measure of the rate of increase of the effective number of the environment subsystems which have recorded the state of the system $\mathcal{S}$.

It is worth noting that either $\mathcal{R}_{\mathcal{I}}$ or $\mathcal{R}_{\mathcal{J}}$ can keep on increasing after the density matrix of the system has lost its off-diagonal terms in the pointer basis and after it can be therefore considered completely decohered. Indeed, direct interaction between the system and the environment is not needed for either $\mathcal{R}$ to change. For example, information about the system inscribed in the primary environment may be communicated to a secondary, tertiary, and more remote environment (which need not interact with the system at all).

It is natural to define objectivity and, therefore, classicality with the help of $\mathcal{R}$. In the limit $\mathcal{R}_{\mathcal{J}} \to \infty$ the information about the preferred states of the system is spread so widely that it can be acquired by many observers simultaneously [11]. Moreover, it already exists in multiple copies, so it can be safely cloned in spite of the no-cloning theorem [20]. Thus, a state of the system redundantly recorded in the environment has all the symptoms of "objective existence". In particular, such well-advertised states can be found out without being disturbed by approximately $\mathcal{R}$ observers acting independently [11,21] (each simply measuring the state of $\sim 1/\mathcal{R}_{\mathcal{J}}$ fraction of the environment).

6 Summary and Conclusions

Information theory offers a useful perspective on the measurement process, on decoherence and, above all, on the definitions of classicality. Discord can be used to characterize the nature of quantum correlations and to distinguish the ones that are classical. The redundancy ratio is a powerful measure of the classicality of states: While a vanishing discord is a necessary condition for classicality of correlations, the redundancy ratio is a direct measure of objective existence of quantum states. Objectivity can be defined operationally as the ability to find what the state is without disturbing it [11,21]. Objective existence of quantum states would make them effectively classical, and was the ultimate goal of the interpretation of quantum theory. We have established it here by investigating einselection from the point of view of information theory and by shifting focus from the system to the environment which is monitoring the system.

These advances clarify some of the interpretational issues which are now a century old. The relation between the epistemological and ontological significance of the quantum state vectors is now apparent: Objective existence of the quantum states is a direct consequence of the redundant records permeating the environment. Epistemology begets ontology!

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References

[1] Zurek, W. H., Phys. Rev. D **24** (1981) 1516-1524
[2] Zurek, W. H., Phys. Rev. D **26** (1982) 1862-1880
[3] Zurek, W. H., Information transfer in quantum measurements, pp. 87-116 in *Quantum Optics, Experimental Gravity, and the Measurement Theory*, P. Meystre and M. O. Scully, eds. (Plenum, New York, 1983)
[4] Schumacher, B., Westmoreland, M., and Wootters, W. K., Phys. Rev. Lett. **76** (1996) 3452-3455; Hall, M. J. H., and O'Rourke, M. J., Quant. Opt. **5** (1993) 161; Halliwell, J. J., Phys. Rev. D **60** (1999) 105031
[5] Joos, E., and Zeh, H. D., Zeits. Phys. B **59** (1985) 229

[6] Caldeira, A. O., and Leggett, A. J., Physica **121A** (1983) 587-616; Phys. Rev. A **31** (1985) 1059; Haake, F., and Reibold, R., Phys. Rev. A **32** (1985) 2462; Grabert, H., Schramm, P., and Ingold., G.-L., Phys. Rev. **168** (1988) 115-207

[7] Zurek, W. H., Reduction of the Wavepacket: How long does it take? presented at a NATO ASI *Frontiers of Nonequilibrium Statistical Mechanics* Santa Fe, June 1984, pp. 145-149 in the proceedings, G. T. Moore and M. O. Scully, eds. (Plenum, New York, 1986); Haake, F., and Walls, D. F., in *Quantum Optics IV*, J. D. Harvey and D. F. Walls, eds. (Springer, Berlin, 1986). Unruh, W. G., and Zurek, W. H., Phys. Rev. D **40** (1989) 1071-1094

[8] Hu, B. L., Paz, J. P., and Zhang, Y., Phys. Rev. D **45** (1992) 2843-2861; Zurek, W. H., Habib, S., and Paz, J. P., Phys. Rev. Lett **70** (1993) 1187-1190; Anglin, J. R., and Zurek, W. H., Phys Rev. D **53** (1996) 7327-7335; Anglin, J. R., Paz, J. P., and Zurek, W. H., Phys. Rev. A **53** (1997) 4041

[9] Brune, M., Hagley, E., Dreyer, J., Maître, X., Maali, A., Wunderlich, C., Raimond, J-M., and Haroche, S., Phys. Rev. Lett. **77** (1996) 4887-4890; Myatt *et al.*, Nature **403** (2000) 269

[10] Giulini, D., Joos, E., Kiefer, C., Kupsch, J., Stamatescu, I.-O., and Zeh, H. D., *Decoherence and the Appearance of a Classical World in Quantum Theory*, (Springer, Berlin, 1996); Blanchard, Ph., Giulini, D., Joos, E., Kiefer, C., and Stamatescu, I.-O., eds, *Decoherence: Theoretical, Experimental, and Conceptual Problems*, (Springer, Berlin, 2000); Paz, J. P., and Zurek, W. H., in *Les Houches Lectures*, in press (2000)

[11] Zurek, W. H., Rev. Mod. Phys., submitted (2000)

[12] Ivanovic I. D., J. Phys. A **14** (1981) 3241-3245; J. Math. Phys. **24** (1983) 1199-1205

[13] Wootters, W. K., and Fields, B. D., Ann. Phys. **191** (1989) 363

[14] Barnett, S. M., and Phoenix, S. J. D., Phys. Rev. A **40** (1989) 2404-2409

[15] Cover, T. M., and Thomas, J. A., *Elements of Information Theory*, (Wiley, New York, 1991)

[16] Zurek, W. H., Progr. Theor. Phys. **89** (1993) 281-302

[17] Zurek, W. H., Habib, S., and Paz, J. P., Phys. Rev. Lett **70** (1993) 1187-1190

[18] Tegmark, M., and Shapiro, H. S., Phys. Rev. E **50** (1994) 2538-2547

[19] Gallis, M. R., Phys. Rev. A **53** (196) 655-660; Paraoanu, Gh.-S., and Scutaru, H., Phys. Lett. A **238** (1998) 219-222

[20] Wootters, W. K., and Zurek, W. H., Nature **299** (1982) 802-802; Dieks, D., Phys. Lett. **92** (1982) 271-272

[21] Zurek, W. H., Phil. Trans. Roy. Soc. Lond. **356** (1998) 1793-1821