Graph Theory Applications EPFL, Spring 2014

# Graded Problem Set 7

Date: 04.04.2014 Due by 18:00 – 11.04.2014

**Instructions :** In the first page, write your name, Sciper number, list of collaborators, and the name of the other (eventual) person which is in your group for the last problem. Then, write the solution for each problem on a separate page. For the last problem you will be required to submit a small report together with some code. Send everything to our email addresses `{marco.mondelli, ruediger.urbanke}@epfl.ch`

**Rules :** As concerns the first 4 problems, you are allowed and encouraged to discuss these problems with your colleagues. However, each of you has to write down her solution in her own words. If you collaborated on a homework, write down the name of your collaborators and your sources in the space below. No points will be deducted for collaborations. But if we find similarities in solutions beyond the listed collaborations we will consider it as cheating. Please note that EPFL has a VERY strict policy on cheating and you might be in BIG trouble. It is simply not worth it. As concerns the last problem (Problem 5), you can work in groups of two people, if you wish. A single report has to be handed in for each of the groups.

**Grading :** Each of the first 4 problems is worth 10 points. The last problem is worth 40 points.

**Problem 1.** *[10 points]* Consider a matrix $A$ whose entries are either 0 or 1. Define $\alpha$ to be the minimum number of lines (i.e., rows or columns) of $A$ that contain all the 1's of $A$. Mark the 1's of $A$ so that each line contains at most a single marked 1. Let $\beta$ the maximum number of 1's that it is possible to mark. Show that $\alpha = \beta$.

**Problem 2.** *[10 points]* Let $G$ be a bipartite graph, with bipartition $(X, Y)$ and with no isolated vertices. Suppose that for every edge $(x, y)$ with one end $x \in X$ and another end $y \in Y$, we have $\deg(x) \geq \deg(y)$. Prove that $G$ has a matching that covers $X$.

**Problem 3.** *[10 points]* Let $G$ be a bipartite graph in which each vertex of $X$ is of odd degree. Suppose that any two vertices of $X$ have an even number of common neighbors. Show that $G$ has a matching covering every vertex of $X$.

**Problem 4.** *[10 points]* Take a standard deck of cards, and deal them out into 13 piles of 4 cards each. Show that it is always possible to select exactly 1 card from each pile, such that the 13 selected cards contain exactly one card of each rank (ace, 2, 3, . . ., queen, king).

**Problem 5.** *[40 points]* In this problem, you will need to use your programming skills. You can use your favorite language, e.g., C, C++, Matlab, Mathematica. In addition, you are allowed to use any general purpose optimization package, but cannot use special graph algorithms like the ones which compute optimal matchings :-)

Attach to the report an electronic version of your code. Comment your code, so that we can understand what you are doing. Add also a README.txt, so that we know how to run it on our machines. Specify how to run the program in a clear way and write if we need to install extra libraries in order to make it work. Write in the report the results of the program and, when requested, a comment/explanation of those results.

In this exercise, we will consider a complete graph $K_n$ on $n$ vertices with independent edge costs $\psi$ having exponential distribution with mean $n$.

1. *[10 points]* Compute the expected minimum edge weight in $K_n$.
2. *[10 points]* Write a program which accepts $n$ as input and outputs the graph $K_n$ with independent and exponential edge costs with mean $n$.
3. *[10 points]* Write a program which accepts $n$ as input. Then, it outputs the minimal cost of a perfect matching in $K_n$.
4. *[10 points]* Try your program of point 3 for $n \in \{8, 14, 20, 26, 32, 38\}$. For each value of $n$, repeat your random experiment $M = 200$ times and plot the average of the minimal cost of a perfect matching divided by $n$ as a function of $n$. Plot also the constant function $f(n) = \pi^2/12$. What do you observe?