Solution to Graded Problem Set 7

Date: 04.04.2014 Due by 18:00 – 11.04.2014

Problem 1. The adjacency matrix of a bipartite graph whose parts have r and s vertices has the form

$$\begin{pmatrix} 0_{r,r} & A \\ A^T & 0_{s,s} \end{pmatrix},$$

where A is an $r \times s$ matrix which is called the bipartite adjacency matrix, and 0 represents the zero matrix. Let the matrix A of the problem be the bipartite adjacency matrix of a graph G. Then, α corresponds to the minimum size of a covering in G, because it counts the lines (i.e., the nodes) which contain all the $1'$s (i.e., the edges) of the matrix (i.e., the bipartite graph). On the other hand, a marking of the 1's s.t. no two of them is on a line corresponds to a matching in G. Hence, β is the maximum size of a matching. As a result, the statement is a consequence of König's theorem.

Problem 2. Suppose the contrary, i.e., there's no matching of X. Then, by Hall's theorem there exists $A \subseteq X$ s.t. $|A| > |N(A)|$. Choose among such sets A one whose cardinality is the smallest. Since there are no isolated vertices, $|N(A)| \geq 1$ and $|A| > 1$. Take an arbitrary $x \in A$. Suppose that there exists no matching covering $A \setminus \{x\}$. Then, $|N(A \setminus \{x\})| < |A \setminus \{x\}|$ contradicting the minimality of A. Therefore, there exists a matching covering $A \setminus \{x\}$ and the following chain of inequalities holds:

$$|N(A)| \geq |N(A \setminus \{x\})| \geq |A \setminus \{x\}| = |A| - 1.$$

Since $|A| > |N(A)|$, we conclude that $|N(A)| = |N(A \setminus \{x\})| = |A| - 1$. As a result,

$$\sum_{v \in A} \deg(v) > \sum_{v \in A \setminus \{x\}} \deg(v) \geq \sum_{v \in N(A \setminus \{x\})} \deg(v) = \sum_{v \in N(A)} \deg(v),$$

where the second inequality comes from the fact that $\deg(x) \geq \deg(y)$ for any $x \in X$ and $y \in Y$ connected by an edge.

Now, note that $\sum_{v \in A} \deg(v)$ counts the edges starting from the set $A \subseteq X$. All these edges will arrive into the set $N(A)$, whose vertices may have some other edge which does not go toward A. Hence, $\sum_{v \in A} \deg(v) \leq \sum_{v \in N(A)} \deg(v)$, which is a contradiction.

Problem 3. For an arbitrary $A \subseteq X$ consider the bipartite adjacency matrix between A and $N(A)$. The idea is to consider this matrix over $\mathbb{F}_2$. Any two vertices of A have an even number of common neighbors and an odd total number of neighbors. Therefore, the inner product between any two rows is 0, which means that the rows of the matrix are pairwise orthogonal.

Since any set of non-zero pairwise orthogonal vectors is linearly independent, the rows are linearly independent, which is possible only if $|A| \leq |N(A)|$, and we are done by Hall's theorem.

Problem 4. Consider a bipartite graph G with bipartition (X, Y), where X is the set of 13 piles and Y is the set of 13 possible ranks. Each pile is connected by an edge with the ranks that appear in it. Clearly, for any k piles, there are at least k ranks appearing in them. Thus, by Hall's theorem, there exists a matching which saturates all elements of X. Since $|X| = |Y|$, such a matching is perfect.

Problem 5. Let $X_1, \ldots, X_n$ be independent exponentially distributed random variables with parameters $\lambda_1, \ldots, \lambda_n$. Then $\min\{X_1, \ldots, X_n\}$ is also exponentially distributed, with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$. This can be seen by considering the complementary cumulative distribution function:

$$
\begin{aligned}
Pr\left(\min\{X_1, \ldots, X_n\} > x\right) &= \Pr\left(X_1 > x \wedge \cdots \wedge X_n > x\right) && (1)\\
&= \prod_{i=1}^{n} \Pr(X_i > x) && (2)\\
&= \prod_{i=1}^{n} \exp(-x\lambda_i) = \exp\left(-x \sum_{i=1}^{n} \lambda_i\right). && (3)
\end{aligned}
$$

As a result, the minimum edge weight in K_n is an exponential random variable with parameter $\binom{n}{2}/n$. Therefore, the expected minimum weight is $n/\binom{n}{2}$.

As concerns the expected minimal cost of a perfect matching in K_n divided by n when the edges are i.i.d. exponential random variables with mean n, such a value converges to $\pi^2/12$ as n grows large. In particular, you should obtain a behavior similar to the one exemplified in Figure 1.

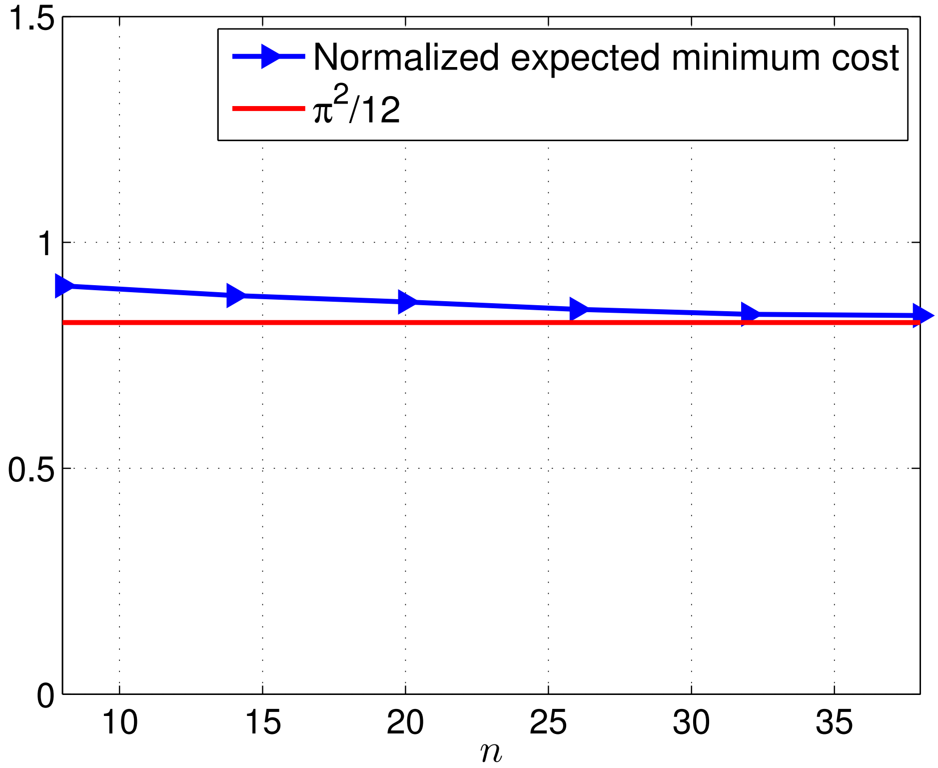

Figure 1: Expected minimal cost of a perfect matching in K_n as a function of n. As n increases, this value quickly converges to $\pi^2/12$.