Graph Theory Applications EPFL, Spring 2014

Problem Set 5

Date: 20.03.2014 Not graded

Problem 1. Show that for any graph G with incidence matrix B and adjacency matrix A,

$$BB^T = A + D,$$

where D denotes a matrix whose diagonal element $(D)_{i,i}$ equals the degree of the vertex i, while the off-diagonal elements $(D)_{i,j}$ with $i \neq j$ are zero.

Problem 2. Is there a graph with degrees 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 7, 7 ?

Problem 3. In a tree that has 100 vertices, how many nodes of degree 10 can be at most?

Problem 4. Given a simple connected graph $G = (V, E)$ with edge costs w_e for each $e \in E$ (assume all edge costs are distinct), prove the following two fundamental properties that are used in all Minimum Spanning Tree (MST) algorithms.

a) (Cut Property) For any proper subset $S \subset V$ of nodes in G, let $e = (u, v)$ be the edge with minimum weight such that $u \in S$ and $v \in V \setminus S$. Show that every MST must contain e.

b) (Cycle Property) Let C be a cycle in G. Let $e = (u, v)$ be the edge with maximum weight in C. Show that e is not in any MST of G.

Problem 5. The telecommunications company *Fraud&Friends* has obtained the contract to build a new infrastructure in Lausanne. We can schematize the network as follows. There are n points that need to be connected and the cost of building a point-to-point cable for each of the possible pairs is known. The telecommunications company will be reimbursed by the commune of Lausanne of the minimum cost required to grant full connectivity to the network.

1) How can the commune of Lausanne compute this cost efficiently?

2) The algorithm of point 1) gives as output a sub-network which grants full connectivity at minimum cost. The algorithm is built by Rudiger and checked by Marco. The CEO of *Fraud& Friends* corrupts Rudiger. In particular, the telecommunications company has already built the network, but wants to get the maximum possible reimbursement from the commune of Lausanne. Marco is very lazy and he will only check that there are no cycles in the sub-network which is given as output by Rudiger's algorithm. How can Rudiger design an algorithm to maximize the reimbursement?